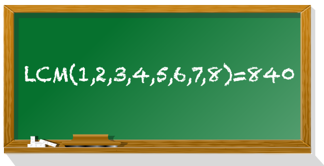

Problem of the Week

Problem E and Solution

Four More

Problem

For each positive integer n, $\text{LCM}(1, 2, \ldots, n)$ is the *least common multiple* of $1, 2, \ldots, n$. That is, the smallest positive integer divisible by each of $1, 2, \ldots, n$.

Determine all positive integers n, with $1 \leq n \leq 100$ such that

$$\text{LCM}(1, 2, \ldots, n) = \text{LCM}(1, 2, \ldots, n + 4)$$

NOTE: In solving this problem, it might be helpful to know that we can calculate the LCM of a set of positive integers by

- determining the prime factorization of each integer in the set,
- determining the list of prime numbers that occur in these prime factorizations,
- determining the highest power of each prime number from this list that occurs in the prime factorizations, and
- multiplying these highest powers together.

For example, $\text{LCM}(1, 2, 3, 4, 5, 6, 7, 8) = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 840$, since the prime factorizations of 2, 3, 4, 5, 6, 7, and 8 are 2, 3, 2^2, 5, $2 \cdot 3$, 7, and 2^3, respectively.

Solution

Since $\text{LCM}(1, 2, \ldots, n)$ is the least common multiple of $1, 2, \ldots, n$ and $\text{LCM}(1, 2, \ldots, n + 4)$ is the least common multiple of $1, 2, \ldots, n, n + 1, n + 2, n + 3, n + 4$, then $\text{LCM}(1, 2, \ldots, n) \neq \text{LCM}(1, 2, \ldots, n + 4)$ if either

(i) there are prime factors that occur in $n + 1$, $n + 2$, $n + 3$, $n + 4$ that don't occur in any of $1, 2, \ldots, n$, or

(ii) there is a higher power of a prime that occurs in the factorizations of one of $n + 1$, $n + 2$, $n + 3$, $n + 4$ that doesn't occur in any of $1, 2, \ldots, n$.

For (i) to occur, consider a prime p that is a divisor of one of $n + 1$, $n + 2$, $n + 3$, $n + 4$, and none of $1, 2, \ldots, n$.. This means that the smallest positive integer that has p as a divisor is one of the integers $n + 1$, $n + 2$, $n + 3$, $n + 4$, which in fact means that this integer equals p. (The smallest multiple of a prime p is $1 \cdot p = p$ itself.)

Thus, for (i) to occur, one of $n+1$, $n+2$, $n+3$, $n+4$ is a prime number.

For (ii) to occur, consider a prime power p^k, where k is an integer >1, that is a divisor of one of $n+1$, $n+2$, $n+3$, $n+4$ and none of $1, 2, \ldots, n$. This means that the smallest positive integer that has p^k as a divisor is one of the integers $n+1$, $n+2$, $n+3$, $n+4$, which in fact means that this integer equals p^k. (The smallest multiple of p^k is p^k itself.)

Therefore, $\text{LCM}(1, 2, \ldots, n) \neq \text{LCM}(1, 2, \ldots, n+4)$ whenever one of $n+1$, $n+2$, $n+3$, $n+4$ is a prime number or a prime power.

In other words, $\text{LCM}(1, 2, \ldots, n) = \text{LCM}(1, 2, \ldots, n+4)$ whenever none of $n+1$, $n+2$, $n+3$, $n+4$ is a prime number or a prime power.

Therefore, we want to determine the positive integers n with $1 \leq n \leq 100$ for which none of $n+1$, $n+2$, $n+3$, $n+4$ is a prime number or a prime power.

The prime numbers less than or equal to 104 are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103$$

The prime powers, with exponent >1, less than or equal to 104 are

$$2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 5^2 = 25,\ 7^2 = 49$$

Therefore, we want to count the positive integers n with $1 \leq n \leq 100$ for which none of $n+1$, $n+2$, $n+3$, $n+4$ appear in the list

$$2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, 41, 43, 47, 49, 53, 59, 61, 64, 67, 71, 73, 79, 81, 83, 89, 97, 101, 103$$

For four consecutive integers to not occur in this list, we need a difference between adjacent numbers to be at least 5.

The values of n that satisfy this condition are $n = 32$, 53, 54, 73, 74, 83, 84, 89, 90, 91, 92. (For example, 54 is a value of n that works since none of 55, 56, 57, 58 appears in the list.)

Therefore, there are 11 values of n with $1 \leq n \leq 100$ for which $\text{LCM}(1, 2, \ldots, n) = \text{LCM}(1, 2, \ldots, n+4)$.