Problem of the Week

Problem E and Solution

Sixty-Four!

Problem

The product $64 \times 63 \times 62 \times \cdots \times 3 \times 2 \times 1$ can be written as 64! and called "64 *factorial*".

In general, the product of the positive integers 1 to m is

$$m! = m \times (m-1) \times (m-2) \times \cdots \times 3 \times 2 \times 1$$

If 64! is divisible by 2025^n, determine the largest positive integer value of n.

Solution

Let $P = 64!$. The prime factorization of 2025 is $3^4 \times 5^2$. We must determine how many times the factors of 3 and 5 are repeated in the factorization of P.

First we count the number of factors of 3 in P by looking at the multiples of 3 from 1 to 64. They are 3, 6, 9, …, 57, 60, and 63. Each of these 21 numbers contains a factor of 3.

Now, each multiple of 9 from 1 to 64 will contain a second factor of 3. These multiples of 9 are 9, 18, 27, 36, 45, 54, and 63. Each of these 7 numbers contains two factors of 3.

Now, each multiple of 27 from 1 to 64 will contain a third factor of 3. These multiples of 27 are 27 and 54. Each of these 2 numbers contains three factors of 3.

There are no higher powers of 3 less than 64. Thus, P has $21 + 7 + 2 = 30$ factors of 3, and so the largest power of 3 that P is divisible by is 3^{30}.

Next we count the number of factors of 5 in P by looking at the multiples of 5 from 1 to 64. They are 5, 10, 15, …, 50, 55, and 60. Each of these 12 numbers contains a factor of 5.

Now, each multiple of 25 from 1 to 64 will contain a second factor of 5. These multiples of 25 are 25 and 50. Each of these 2 numbers contains two factors of 5.

There are no higher powers of 5 less than 64. Thus, P has $12 + 2 = 14$ factors of 5, and so the largest power of 5 that P is divisible by is 5^{14}.

Thus, P is divisible by $3^{30} \times 5^{14}$.

$$3^{30} \times 5^{14} = 3^{28} \times 3^2 \times 5^{14} = \left(3^4 \times 5^2\right)^7 \times 3^2 = 2025^7 \times 3^2$$

Thus, P is divisible by 2025^7, and 7 is the largest value of n.