Problem of the Week

Problem E and Solution

A Tale of Two Towns

Problem

Two towns, Centreville and Middletown, had the same population at the end 2022.

The population of Centreville decreased by 2.5% from the end of 2022 to the end of 2023. Then, the population increased by 8.4% from the end of 2023 to the end of 2024.

The population of Middletown increased by $r\%$, where $r > 0$, from the end of 2022 to the end of 2023. Then, the population of Middletown increased by $(r+2)\%$ from the end of 2023 to the end of 2024.

Surprisingly, the populations of both towns were the same again at the end of 2024. Determine the value of r, rounded to the nearest tenth.

Solution

Let p be the population of Centreville at the end of 2022. Since Centreville and Middletown have the same population size at the end of 2022, then p is also the population of Middletown at the end of 2022.

The population of Centreville decreased by 2.5% in 2023, so the population at the end of 2023 was

$$p - \frac{2.5}{100}p = \left(1 - \frac{2.5}{100}\right)p = 0.975p$$

The population of Centreville then increased by 8.4% during 2024, so the population at the end of 2024 was

$$0.975p + \left(\frac{8.4}{100}\right)(0.975p) = \left(1 + \frac{8.4}{100}\right)(0.975p) = 1.084(0.975p) = 1.0569p$$

The population of Middletown increased by $r\%$ in 2023, so the population at the end of 2023 was

$$p + \frac{r}{100}p = \left(1 + \frac{r}{100}\right)p$$

The population of Middletown then increased by $(r+2)\%$ during 2024, so the population at the end of 2024 was

$$\left(1 + \frac{r}{100}\right)p + \frac{r+2}{100}\left(1 + \frac{r}{100}\right)p = \left(1 + \frac{r}{100}\right)\left(1 + \frac{r+2}{100}\right)p$$

Since the populations of Centreville and Middletown are equal at the end of 2024, we have

$$\left(1 + \frac{r}{100}\right)\left(1 + \frac{r+2}{100}\right)p = 1.0569p$$

Dividing both sides by $p > 0$ and multiplying both sides by 10 000 to clear fractions, we have $(100 + r)(102 + r) = 10\,569$. Thus, $10\,200 + 202r + r^2 = 10\,569$, and so $r^2 + 202r - 369 = 0$.

Using the quadratic formula, $r = \dfrac{-202 \pm \sqrt{202^2 - 4(-369)}}{2} \approx 1.8, -203.8$.

Since $r > 0$, we have $r \approx 1.8\%$, correct to one decimal place.