



Problem of the Week

Problem E and Solution

Sums with Multiples of Three

Problem

The set $\{3, 6, 9, 12, 15, \dots, 2022, 2025\}$ contains all of the multiples of three from 3 to 2025. Three distinct numbers are chosen from the set to form a sum. How many different sums can be formed?

Solution

Since the set includes every multiple of three from 3 to 2025, then there are $2025 \div 3 = 675$ numbers in the set. Each number is of the form $3n$, for $n = 1, 2, 3, \dots, 675$. The required sum is $3a + 3b + 3c$, where a, b , and c are three distinct numbers chosen from $\{1, 2, 3, \dots, 675\}$. But $3a + 3b + 3c = 3(a + b + c)$. Thus, we can reduce the problem to the question of, “How many distinct integers can be formed by adding three numbers from the set $\{1, 2, 3, \dots, 675\}$?”

The smallest sum is $1 + 2 + 3 = 6$ and the largest sum is $673 + 674 + 675 = 2022$. It is possible to get every number in between 6 and 2022 by:

- (a) increasing the sum by 1 by replacing one of the three numbers with a number that is 1 larger or,
- (b) decreasing the sum by 1 replacing one of the three numbers with a number that is 1 smaller.

Therefore, all of the integers from 6 to 2022 inclusive can be formed. There are 2022 integers from 1 to 2022, inclusive. But this includes the five integers from 1 to 5. So, there are $2022 - 5 = 2017$ integers from 6 to 2022. Thus, 2017 different sums can be formed.

This answer, 2017, is the answer to the original problem as well. If $a + b + c = 6$, then $3(a + b + c) = 18$. This is the smallest integer that is the sum of the three smallest numbers, 3, 6 and 9, from the original set. If $a + b + c = 2022$, then $3(a + b + c) = 6066$. This is the largest integer that is the sum of the three largest numbers, 2019, 2022, and 2025, from the original set. Then every multiple of three from 18 to 6066 can be generated by adding three different numbers from the original set. (There are 2017 multiples of three from 18 to 6066, inclusive. And each of these can be obtained by adding three distinct numbers from the original set.)