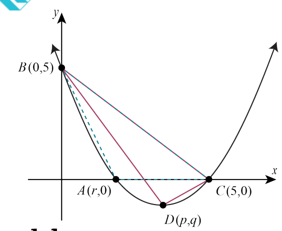


Problem of the Week

Problem E and Solution

No Parabolema to Find the Area!

Problem

A parabola intersects the y-axis at $B(0,5)$, and intersects the x-axis at $C(5,0)$ and at $A(r,0)$, where $0 < r < 5$. The area of $\triangle ABC$ is 5 units^2.

If $D(p,q)$ is the vertex of the parabola, then determine the area of $\triangle DBC$.

Solution

The height of $\triangle ABC$ is the distance from the x-axis to $B(0,5)$, which is 5 units. The base is $AC = 5 - r$. Since the area of $\triangle ABC$ is 5, using the formula for the area of a triangle, we have $\frac{(5-r)(5)}{2} = 5$. Then $5 - r = 2$ and $r = 3$ follows. Thus, the coordinates of A are $(3,0)$.

The axis of symmetry of the parabola is a vertical line through the midpoint of AC, which is $(4,0)$. It follows that the x-coordinate of the vertex is $p = 4$. Therefore, the vertex is $D(4,q)$.

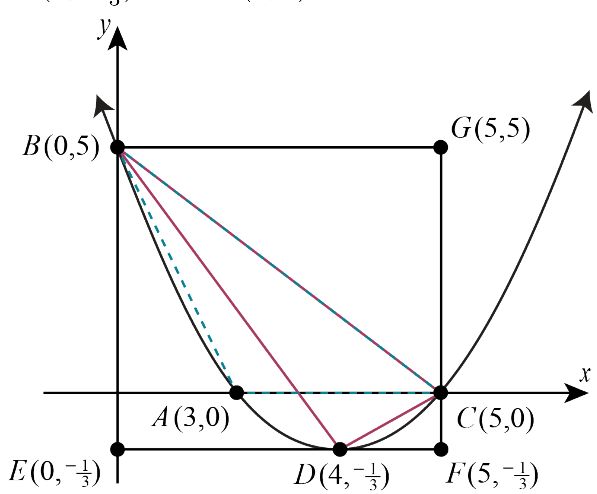
Since the two x-intercepts of the parabola are 3 and 5, the equation of the parabola in factored form can be written as $y = a(x-3)(x-5)$. Since the parabola passes through $B(0,5)$, we can solve for a by substituting $x = 0$ and $y = 5$ into $y = a(x-3)(x-5)$. This leads to $a = \frac{1}{3}$ and thus the parabola has equation $y = \frac{1}{3}(x-3)(x-5)$.

To determine q, the y-coordinate of D, we substitute $x = 4$, $y = q$ into $y = \frac{1}{3}(x-3)(x-5)$. Then $q = \frac{1}{3}(4-3)(4-5) = -\frac{1}{3}$. Therefore, D has coordinates $(4, -\frac{1}{3})$.

From here, we proceed with two different solutions to determine the area of $\triangle DBC$.

Solution 1

Consider points $E(0, -\frac{1}{3})$, $F(5, -\frac{1}{3})$, and $G(5,5)$, and draw in $BGFE$.



Since B and G have the same y-coordinate, BG is a horizontal line. Since G and F both have x-coordinate 5, GF is a vertical line which passes through C. Since E and F both have y-coordinate $-\frac{1}{3}$, EF is a horizontal line which passes through D. Since B and E have the same x-coordinate, BE is a vertical line. Thus, $BGFE$ is a rectangle that encloses $\triangle DBC$, and we have

$$\text{area } \triangle DBC = \text{area } BGFE - \text{area } \triangle BGC - \text{area } \triangle DFC - \text{area } \triangle BED$$

In rectangle $BGFE$, $BG = 5 - 0 = 5$ and $BE = 5 - (-\frac{1}{3}) = \frac{16}{3}$. The area of rectangle $BGFE = BG \times BE = 5 \times \frac{16}{3} = \frac{80}{3}$ units^2.

Since $BGFE$ is a rectangle, $\triangle BGC$ is right-angled at G. Since $BG = 5$ and $GC = 5 - 0 = 5$, the area of $\triangle BGC = \frac{BG \times GC}{2} = \frac{5\times 5}{2} = \frac{25}{2}$ units^2.

Since $BGFE$ is a rectangle, $\triangle DFC$ is right-angled at F. Since $CF = 0 - (-\frac{1}{3}) = \frac{1}{3}$ and $DF = 5 - 4 = 1$, the area of $\triangle DFC = \frac{CF \times DF}{2} = \frac{\frac{1}{3}\times 1}{2} = \frac{1}{6}$ units^2.

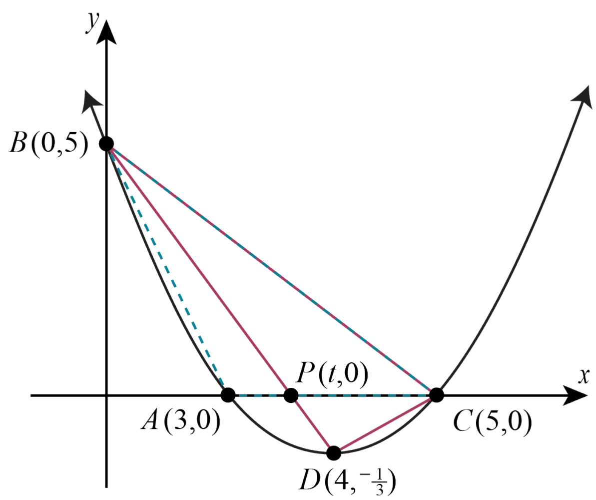
Since $BGFE$ is a rectangle, $\triangle BED$ is right-angled at E. Since $BE = \frac{16}{3}$ and $ED = 4 - 0 = 4$, the area of $\triangle BED = \frac{BE \times ED}{2} = \frac{\frac{16}{3}\times 4}{2} = \frac{32}{3}$ units^2.

Thus,

$$\begin{aligned}\text{area } \triangle DBC &= \text{area } BGFE - \text{area } \triangle BGC - \text{area } \triangle DFC - \text{area } \triangle BED \\ &= \tfrac{80}{3} - \tfrac{25}{2} - \tfrac{1}{6} - \tfrac{32}{3} \\ &= \tfrac{10}{3} \text{ units}^2\end{aligned}$$

Solution 2

Let $P(t, 0)$ be the point where the line through B and D crosses the x-axis. We will determine the equation of the line that passes through B, P, and D.



Since the line passes through $B(0, 5)$ and $D(4, -\frac{1}{3})$, the slope of the line is $\frac{5+\frac{1}{3}}{0-4} = \frac{\frac{16}{3}}{-4} = -\frac{4}{3}$.

The y-intercept of the line is 5. Therefore, the equation of the line through B, P, and D is $y = -\frac{4}{3}x + 5$.

To determine t, the x-coordinate of P we substitute $x = t$ and $y = 0$ into $y = -\frac{4}{3}x + 5$, the equation of the line. Thus, $0 = -\frac{4}{3}t + 5$, and $4t = 15$ or $t = \frac{15}{4}$ follows.

In $\triangle BPC$, the height is the perpendicular distance from the x-axis to point B, which is 5. The base is $PC = 5 - \frac{15}{4} = \frac{5}{4}$. Thus, the area of $\triangle BPC = \frac{\frac{5}{4}\times 5}{2} = \frac{25}{8}$ units^2.

In $\triangle DPC$, the height is the perpendicular distance from the x-axis to point D, which is $\frac{1}{3}$. The base is $PC = 5 - \frac{15}{4} = \frac{5}{4}$. Thus, the area of $\triangle DPC = \frac{\frac{5}{4}\times\frac{1}{3}}{2} = \frac{5}{24}$ units^2.

Therefore, the area of $\triangle DBC = \text{area } \triangle BPC + \text{area } \triangle DPC = \frac{25}{8} + \frac{5}{24} = \frac{10}{3}$ units^2.