Problem of the Week

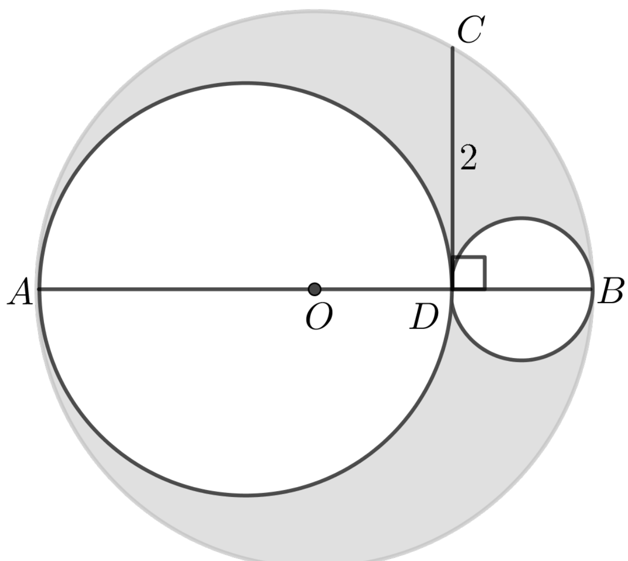
Problem E and Solution

Embedded Circles

Problem

A circle with centre O has diameter AB. A line segment is drawn from a point C on the circumference of the circle to D on OB such that $CD \perp OB$ and $CD = 2$ units. Two circles are drawn on AB. One has diameter AD and the other has diameter DB.

Determine the area of the shaded region. That is, determine the area inside the circle centred at O but outside of the circle with diameter AD and outside of the circle with diameter DB.



Solution

Let the radius of the circle with diameter DB be r. Then $DB = 2r$. Let the radius of the circle with diameter AD be R. Then $AD = 2R$.
Also, $AB = AD + DB = 2R + 2r$, and so the radius of the circle with centre O is $R + r$.

It follows that the area of the circle with diameter AD is πR^2, the area of the circle with diameter DB is πr^2, and the area of the circle with centre O is $\pi(R + r)^2$.

To determine the shaded area, we calculate the area of the circle with centre O and subtract the area of the circle with diameter AD and the area of the circle with diameter DB. That is,

$$\begin{aligned}\text{Shaded Area} &= \pi(R+r)^2 - \pi R^2 - \pi r^2 \\ &= \pi(R^2 + 2Rr + r^2) - \pi R^2 - \pi r^2 \\ &= \pi R^2 + 2\pi Rr + \pi r^2 - \pi R^2 - \pi r^2 \\ &= 2\pi Rr\end{aligned}$$

Join A to C and C to B. Since AB is a diameter and $\angle ACB$ is inscribed in a circle by that diameter, we know that $\angle ACB = 90^\circ$.

Since $CD \perp OB$, then $\angle ODC = \angle ADC = \angle BDC = 90^\circ$. We will use the Pythagorean Theorem in the three triangles $\triangle ADC$, $\triangle BDC$, and $\triangle ACB$, to establish a relationship between R and r.

In $\triangle ADC$, $AC^2 = AD^2 + CD^2 = (2R)^2 + 2^2 = 4R^2 + 4$.

In $\triangle BDC$, $BC^2 = DB^2 + CD^2 = (2r)^2 + 2^2 = 4r^2 + 4$.

In $\triangle ACB$, $AB^2 = AC^2 + BC^2 = (4R^2 + 4) + (4r^2 + 4) = 4R^2 + 4r^2 + 8$.

But $AB^2 = (AD + DB)^2 = (2R + 2r)^2 = 4R^2 + 8Rr + 4r^2$.
Therefore, $4R^2 + 8Rr + 4r^2 = 4R^2 + 4r^2 + 8$ and $8Rr = 8$ or $Rr = 1$ follows.

Thus, the shaded area is equal to $2\pi Rr = 2\pi(1) = 2\pi$ units^2.

NOTE: The relationship $Rr = 1$ could also be established using similar triangles as follows:

In $\triangle ADC$, $\angle CAD + \angle ACD = 90^\circ$. Also, since $\angle ACB = 90^\circ$, $\angle ACD + \angle DCB = 90^\circ$.

Thus, $\angle CAD + \angle ACD = \angle ACD + \angle DCB$, which simplifies to $\angle CAD = \angle DCB$.

Now $\angle CAD = \angle DCB$ and $\angle CDA = \angle CDB = 90^\circ$.

Therefore, $\triangle ADC \sim \triangle CDB$ by Angle Angle Angle (AAA) similarity.

From triangle similarity, $\dfrac{AD}{CD} = \dfrac{CD}{DB}$, and so $\dfrac{2R}{2} = \dfrac{2}{2r}$ and $Rr = 1$ follows.