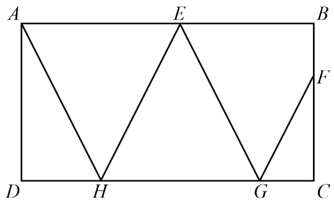
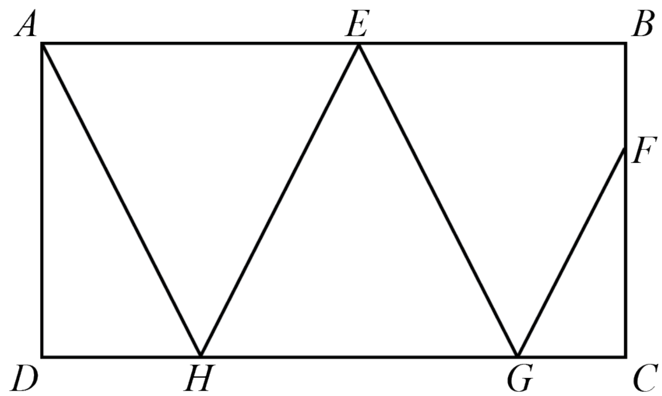


Problem of the Week
Problem E and Solution
Zigzagged

Problem

A fence is to be constructed in a zigzag pattern inside a rectangular field, as shown.

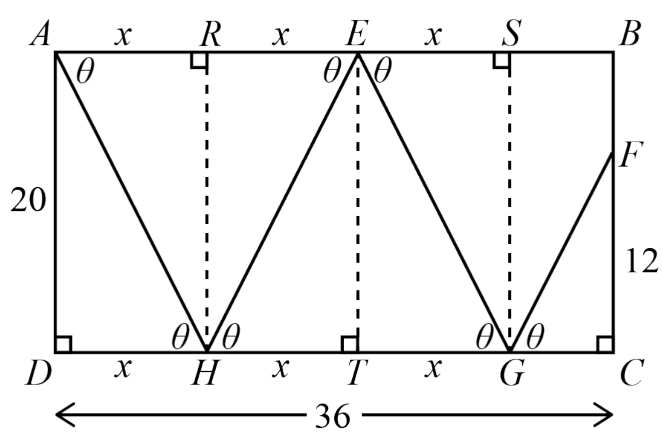


The fence will be constructed so that $\angle AHD = \angle EHG$, $\angle AEH = \angle BEG$, $\angle EGH = \angle FGC$, and $CF = 12$ m. If $AB = 36$ m and $AD = 20$ m, determine the total length of fencing required. That is, determine the value of $AH + EH + EG + FG$.

Solution

Since $ABCD$ is a rectangle, then $AB||CD$. Then $\angle EAH = \angle AHD$, $\angle AEH = \angle EHG$, and $\angle BEG = \angle EGH$. Since $\angle AHD = \angle EHG$, $\angle AEH = \angle BEG$, and $\angle EGH = \angle FGC$, it follows that $\angle EAH = \angle AHD = \angle EHG = \angle AEH = \angle BEG = \angle EGH = \angle FGC = \theta$.

Let R and S be on AB such that RH and SG are perpendicular to AB. Let T be on CD such that ET is perpendicular to CD. Then $\triangle ADH$, $\triangle ARH$, $\triangle ERH$, $\triangle HTE$, $\triangle GTE$, and $\triangle ESG$ all have equal angles and a height of 20 m, so they are all congruent. Let $AR = RE = ES = DH = HT = TG = x$.



Since $\triangle ADH$ and $\triangle FCG$ have equal angles, it follows that they are similar. Then

$$\begin{aligned} \frac{GC}{FC} &= \frac{DH}{AD} \\ \frac{GC}{12} &= \frac{x}{20} \\ GC &= \frac{x}{20} \times 12 = \frac{3x}{5} \end{aligned}$$

Since $DH + HT + TH + GC = 36$, then $x + x + x + \frac{3x}{5} = 36$. Then $\frac{18x}{5} = 36$, so $x = 10$. Then $GC = \frac{3(10)}{5} = 6$. By the Pythagorean Theorem in $\triangle FCG$,

$$\begin{aligned} FG^2 &= FC^2 + GC^2 \\ &= 12^2 + 6^2 \\ &= 180 \end{aligned}$$

Then $FG = \sqrt{180} = 6\sqrt{5}$, since $FG > 0$.

By the Pythagorean Theorem in $\triangle ADH$,

$$\begin{aligned} AH^2 &= AD^2 + DH^2 \\ &= 20^2 + 10^2 \\ &= 500 \end{aligned}$$

Then $AH = \sqrt{500} = 10\sqrt{5}$, since $AH > 0$.

Since $\triangle ADH$, $\triangle HTE$, and $\triangle ETG$ are congruent, it follows that $AH = EH = EG = 10\sqrt{5}$. The total length of fencing required is equal to $AH + EH + EG + FG$, which is $10\sqrt{5} + 10\sqrt{5} + 10\sqrt{5} + 6\sqrt{5} = 36\sqrt{5}$ m.