Problem of the Week

Problems and Solutions 2024-2025

Problem E

Grade 11/12

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

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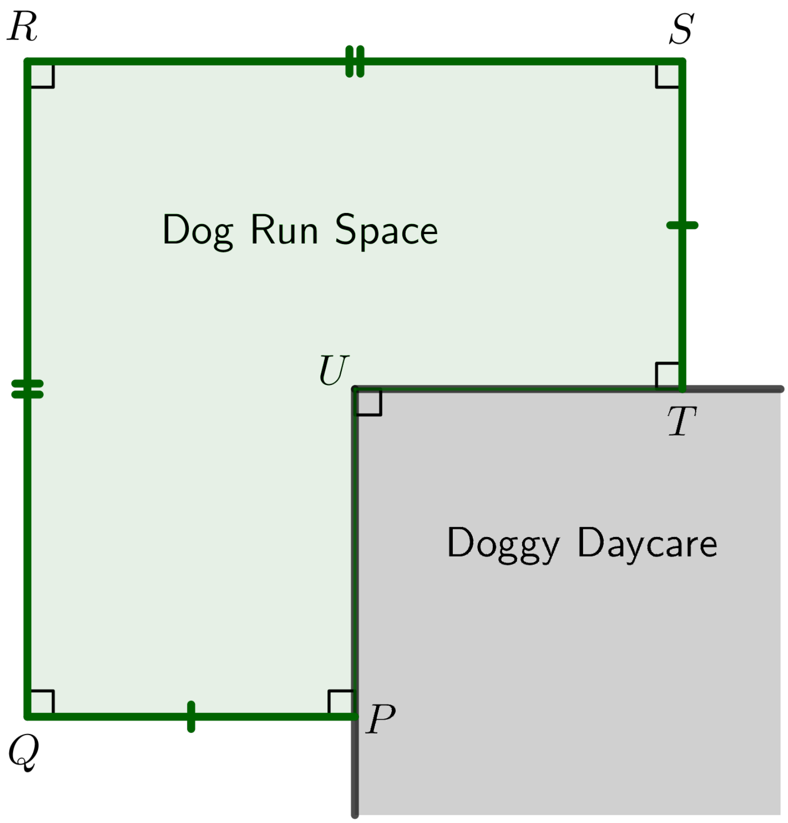
Problem of the Week

Problem E

Run Dog Run

At POTW Doggy Care, there is a need for a new outdoor dog run space. The layout of the dog run space is represented by $PQRSTU$ in the diagram below.



The lengths of the two longer sides, QR and RS, are to be the same, and the lengths of the two shorter sides, PQ and ST, are to be the same. There will be right angles at each corner.

The dog run space is to be built using a fence along PQ, QR, RS, and ST, and using the walls of the daycare along PU and TU. The total fencing to be used is 30 m. Determine the dimensions of the dog run space that will give the maximum area for the dog run.

Problem of the Week
Problem E and Solution
Run Dog Run

Problem

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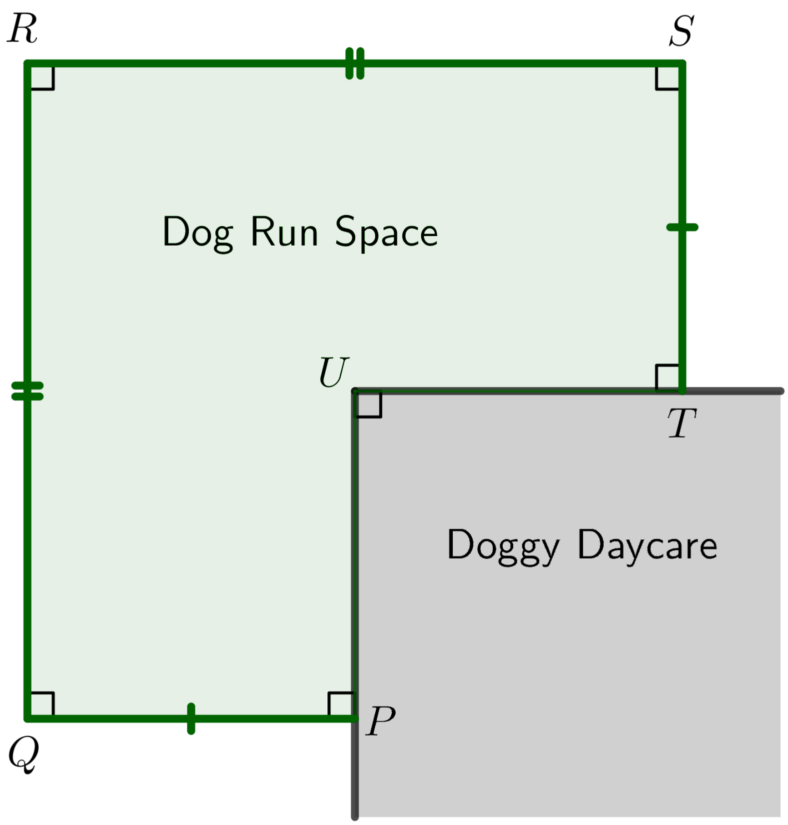

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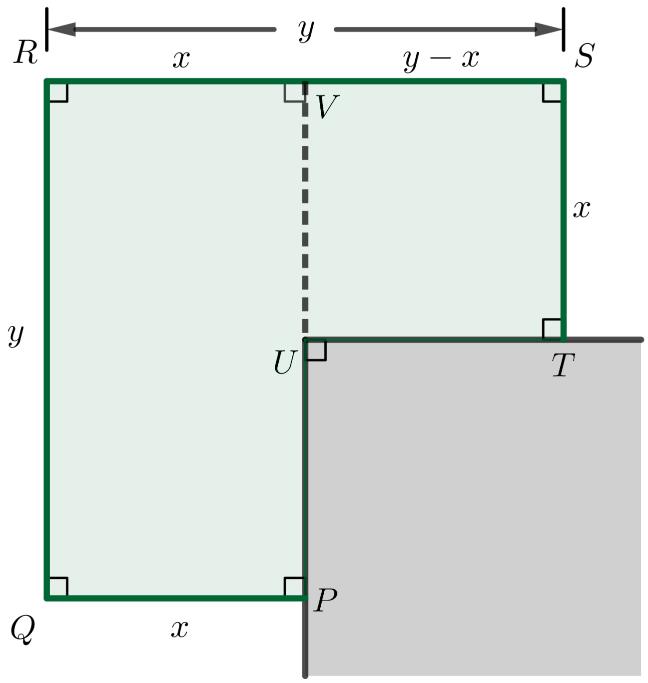
Solution

Extend PU to RS, letting the intersection point be V. Then $PV \perp RS$.

Let x represent the lengths, in metres, of both PQ and ST. Let y represent the lengths, in metres, of both QR and RS. Since $PQRV$ is a rectangle, $RV = PQ = x$ and $VS = RS - RV = y - x$.



The total length of fencing from P to Q to R to S to T is

$$PQ + QR + RS + ST = x + y + y + x = 2x + 2y$$

Since the total amount of fencing used is 30 m, we have $2x + 2y = 30$. Thus, $x + y = 15$ and $y = 15 - x$.

$$\begin{aligned}\text{Area of dog run} &= \text{Area } PQRV + \text{Area } VSTU \\ &= QR \times RV + VS \times ST \\ &= yx + (y - x)x \\ &= 2xy - x^2\end{aligned}$$

Substituting $y = 15 - x$, this becomes

$$\begin{aligned}\text{Area of dog run} &= 2x(15 - x) - x^2 \\ &= 30x - 2x^2 - x^2 \\ &= -3x^2 + 30x\end{aligned}$$

Completing the square, we have

$$\begin{aligned}\text{Area of dog run} &= -3(x^2 - 10x) \\ &= -3(x^2 - 10x + 5^2 - 5^2) \\ &= -3(x^2 - 10x + 25) + 75 \\ &= -3(x - 5)^2 + 75\end{aligned}$$

This is the equation of a parabola which opens down from a vertex of $(5, 75)$. Thus, the maximum area is 75 m^2, and occurs when $x = 5$ m. When $x = 5$, we have $y = 15 - x = 15 - 5 = 10$ m.

Therefore, if $QR = RS = 10$ m and $PQ = ST = 5$ m, this gives a maximum area of 75 m^2.

Problem of the Week

Problem E

Summing up a Sequence 2

The first term in a sequence is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

Shweta writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. How many different possible values of n are there?

24, 12, 6, ...

24, 12, 6, ...

Problem of the Week
Problem E and Solution
Summing up a Sequence 2

Problem

The first term in a sequence is 24. We can determine the next terms in the sequence as follows:

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By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

Shweta writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. How many different possible values of n are there?

Solution

We will begin by finding more terms in the sequence. The first 14 terms of the sequence are 24, 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1.

If we continue the sequence, we will see that the terms 4, 2, and 1 will continue to repeat. Now we want to find the smallest and largest possible values of n so that the sum of the terms in the sequence from term 1 to term n is a four-digit number. We will start by finding the smallest possible value of n.

The sum of the first 8 terms is $24 + 12 + 6 + 3 + 10 + 5 + 16 + 8 = 84$. The sum of the repeating numbers is $4 + 2 + 1 = 7$. We want to determine the number of groups of repeating numbers. Let this be g. Suppose $84 + 7g = 1000$. Solving this gives $7g = 916$, so $g \approx 130.857$.

If $g = 130$, then the sum of the terms in the sequence is $84 + 7 \times 130 = 994$. This sequence contains the first 8 terms, plus 130 groups of the three repeating numbers. Therefore there are a total of $8 + 3 \times 130 = 398$ terms.

The 399^{th} term in the sequence will be 4, so the sum of the first 399 terms will be $994 + 4 = 998$.

The 400^{th} term in the sequence will be 2, so the sum of the first 400 terms will be $998 + 2 = 1000$. This is the smallest possible four-digit number, so the smallest possible value of n is 400.

Now we will find the largest possible value of n. Using a similar approach, let g be the number of groups of repeating numbers. Suppose $84 + 7g = 9999$. Solving this gives $g \approx 1416.429$.

If $g = 1416$, then the sum of the terms in the sequence is $84 + 7 \times 1416 = 9996$. This sequence contains the first 8 terms, plus 1416 groups of the three repeating numbers. Therefore there are a total of $8 + 3 \times 1416 = 4256$ terms.

The 4257^{th} term in the sequence will be 4, so the sum of the first 4257 terms will be $9996 + 4 = 10\,000$. Since this is not a four-digit number, the largest possible value of n is 4256.

So n can be any positive integer between 400 and 4256, inclusive. This is a total of $4256 - 400 + 1 = 3857$ possible values.

Extension:

In 1937, the mathematician Lothat Collatz wondered if any sequence whose terms after the first are determined in this way would always eventually reach the number 1, regardless of which number you started with. This problem is actually still unsolved today and is called the Collatz Conjecture.

Problem of the Week
Problem E
A Geometric Problem

The first term in a geometric sequence is a, the second term is b, and the third term is c. The three terms have a sum of 158 and a product of 74 088.

Determine all possible ordered triples (a, b, c).

$$t_n = ar^{n-1}$$

Note: The general term of a geometric sequence can be written as $t_n = ar^{n-1}$, where a is first term of the sequence, r is the common ratio between terms, and t_n is the n^{th} term.

Problem of the Week
Problem E and Solution
A Geometric Problem

$t_n = ar^{n-1}$

Problem

The first term in a geometric sequence is a, the second term is b, and the third term is c. The three terms have a sum of 158 and a product of 74 088.

Determine all possible ordered triples (a, b, c).

Solution

Let r be the common ratio of the geometric sequence. Since a is the first term of the sequence, then $b = ar$ and $c = ar^2$.

We are given that $abc = 74\,088$. Thus, $a(ar)(ar^2) = a^3r^3 = (ar)^3 = 74\,088$. Therefore, $ar = 42$. Since $b = ar$, we have $b = 42$.

Now, $a + b + c = 158$ becomes $a + 42 + c = 158$, or $a + c = 116$.

Since $b = ar$, then $42 = ar$, or $r = \dfrac{42}{a}$ (since the product of a, b, and c is not zero, we know $a \neq 0$).

Therefore, $c = ar^2 = a\left(\dfrac{42}{a}\right)^2 = a\left(\dfrac{1764}{a^2}\right) = \dfrac{1764}{a}$.

Substituting $c = \dfrac{1764}{a}$ into $a + c = 116$, we have

$$\begin{aligned} a + \frac{1764}{a} &= 116 \\ a^2 + 1764 &= 116a \\ a^2 - 116a + 1764 &= 0 \\ (a-18)(a-98) &= 0 \end{aligned}$$

Therefore, $a = 18$ or $a = 98$.

When $a = 18$, then $r = \frac{42}{18} = \frac{7}{3}$, and one ordered triple is (18, 42, 98). Indeed, we can check that $18 + 42 + 98 = 158$ and $(18)(42)(98) = 74\,088$.

When $a = 98$, then $r = \frac{42}{98} = \frac{3}{7}$, and one ordered triple is (98, 42, 18). Indeed, we can check that $98 + 42 + 18 = 158$ and $(98)(42)(18) = 74\,088$.

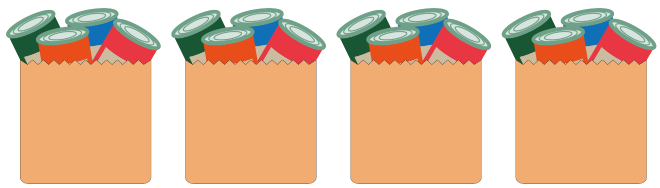
In conclusion there are two ordered triples that satisfy the conditions of the problem. They are (18, 42, 98) and (98, 42, 18).

Problem of the Week

Problem E

Free Time

Pearl coordinates volunteers to help collect and sort donations at the food bank on Mondays, Wednesdays, and Fridays. Of her volunteers, 50% are available on Mondays, 80% are available on Wednesdays, and 90% are available on Fridays. A total of 18 volunteers are available on all three days, and all other volunteers are available on exactly two of the three days. How many volunteers are there in total?



Problem of the Week

Problem E and Solution

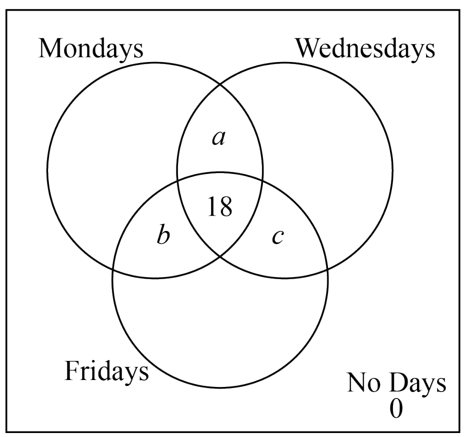
Free Time

Problem

Pearl coordinates volunteers to help collect and sort donations at the food bank on Mondays, Wednesdays, and Fridays. Of her volunteers, 50% are available on Mondays, 80% are available on Wednesdays, and 90% are available on Fridays. A total of 18 volunteers are available on all three days, and all other volunteers are available on exactly two of the three days. How many volunteers are there in total?

Solution

Let a be the number of volunteers who are available on Mondays and Wednesdays, but not Fridays. Let b be the number of volunteers who are available on Mondays and Fridays, but not Wednesdays. Let c be the number of volunteers who are available on Wednesdays and Fridays, but not Mondays. We note that none of the volunteers are available on zero or only one of the three days, and that 18 volunteers are available on all three days. We summarize this information in the following Venn diagram.



Let n be the total number of volunteers. Then, $n = a + b + c + 18$. From the given information,

- 50% of the volunteers are available on Mondays, so $0.5n = a + b + 18$.
- 80% of the volunteers are available on Wednesdays, so $0.8n = a + c + 18$.
- 90% of the volunteers are available on Fridays, so $0.9n = b + c + 18$.

Since $n = a + b + c + 18$, it follows that $2n = 2a + 2b + 2c + 36$. Then,

$$\begin{aligned} 2n &= 2a + 2b + 2c + 36 \\ &= (a + b + 18) + (a + c + 18) + (b + c) \\ &= 0.5n + 0.8n + 0.9n - 18 \\ &= 2.2n - 18 \\ 18 &= 0.2n \\ n &= 90 \end{aligned}$$

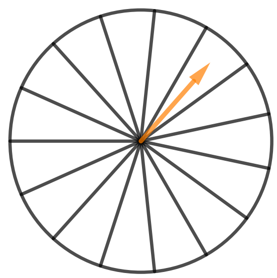
Thus, there are 90 volunteers in total.

Problem of the Week

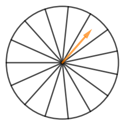
Problem E

Spinning Red

A spinner is divided into 15 equal sections. Each section is coloured either red or yellow. An arrow is attached to the centre of the spinner. Jamal spins the arrow two times. If there is a 64% chance of landing on red in at least one of the two spins, how many red sections are there?



NOTE: You may use the following fact from probability theory: If the probability of event A occurring is a, the probability of event B occurring is b, and the events are not dependent on each other, then the probability of both events occurring is $a \times b$.



Problem of the Week
Problem E and Solution
Spinning Red

Problem

A spinner is divided into 15 equal sections. Each section is coloured either red or yellow. An arrow is attached to the centre of the spinner. Jamal spins the arrow two times. If there is a 64% chance of landing on red in at least one of the two spins, how many red sections are there?

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Solution
Solution 1

Let n be the number of yellow sections. Therefore, for each spin, the probability the spinner lands on a yellow section (and therefore does not land on a red section) is $\frac{n}{15}$. Since the result of each spin does not depend on the previous spin,

$$\begin{aligned} P(\text{not red on either spin}) &= P(\text{not red on first spin}) \times P(\text{not red on second spin}) \\ &= \frac{n}{15} \times \frac{n}{15} \\ &= \left(\frac{n}{15}\right)^2 \end{aligned}$$

The probability of landing on at least one red in two spins is 0.64. So, the probability of not landing on red in either of the two spins is $1 - 0.64 = 0.36$. That is,

$$\left(\frac{n}{15}\right)^2 = 0.36$$

Since $n \geq 0$, this simplifies to $\frac{n}{15} = 0.6$, and so $n = 9$.

Since n is the number of yellow sections, there are $15 - 9 = 6$ red sections.

Solution 2

Let r be the number of red sections. Therefore, the number of yellow sections is $15 - r$. Also, for each spin, the probability the spinner lands on a red section is $\frac{r}{15}$ and the probability the spinner lands on a yellow section is $\frac{15-r}{15}$.

If Jamal lands on red in at least one spin, then he may land red on the first spin only, red on the second spin only, or red on both spins.

If Jamal lands on red on his first spin only, then his second spin must land on yellow. Since the results of each spin do not depend on each other, the probability that he spins red on his first spin and yellow on his second spin is $\frac{r}{15} \times \frac{15-r}{15} = \frac{r(15-r)}{15^2}$.

If Jamal lands on red on his second spin only, then his first spin must land on yellow. Thus, the probability Jamal spins yellow on his first spin and red on his second spin is $\frac{15-r}{15} \times \frac{r}{15} = \frac{r(15-r)}{15^2}$.

The probability that Jamal lands on red on his first spin and again on his second spin is $\frac{r}{15} \times \frac{r}{15} = \frac{r^2}{15^2}$.

The probability of getting at least one red in the two spins is equal to the probability that he lands red on his first spin only, plus the probability that he lands red on his second spin only, plus the probability that he lands red on both spins. That is,

$$\frac{r(15-r)}{15^2} + \frac{r(15-r)}{15^2} + \frac{r^2}{15^2} = 0.64$$

Multiplying both sides by 15^2 gives

$$r(15-r) + r(15-r) + r^2 = 144$$

This simplifies to $r^2 - 30r + 144 = 0$. By factoring, we obtain $(r-6)(r-24) = 0$. Therefore, $r = 6$ or $r = 24$.

Since the spinner has only 15 sections, we must have $r \leq 15$. Thus, the only solution is $r = 6$. That is, there are 6 red sections.

Solution 3

Let r be the number of red sections. Therefore, the number of yellow sections is $15 - r$. Also, for each spin, the probability the spinner lands on a red section is $\frac{r}{15}$ and the probability the spinner lands on a yellow section is $\frac{15-r}{15}$.

If Jamal lands on red in at least one spin, then the first red occurs on his first spin or on his second spin.

If Jamal lands on the first red on his first spin, then on his first spin he spins a red, and on his second spin he spins any colour. Since the results of each spin do not depend on each other, the probability that the first red occurs on his first spin is $\frac{r}{15} \times \frac{15}{15} = \frac{r}{15}$.

If Jamal lands on the first red on the second spin, then on his first spin he spins a yellow, and on his second spin he spins a red. The probability of this is $\frac{15-r}{15} \times \frac{r}{15} = \frac{(15-r)r}{15^2}$.

The probability of spinning at least one red on the two spins is equal to the probability that he lands the first red on his first spin, plus the probability that he lands the first red on his second spin. That is,

$$\begin{aligned} \frac{r}{15} + \frac{(15-r)r}{15^2} &= 0.64 \\ 15r + (15-r)r &= 144 \\ r^2 - 30r + 144 &= 0 \end{aligned}$$

Using the quadratic formula, we find $r = \dfrac{30 \pm \sqrt{30^2 - 4(1)(144)}}{2} = \dfrac{30 \pm 18}{2} = 24, 6$.

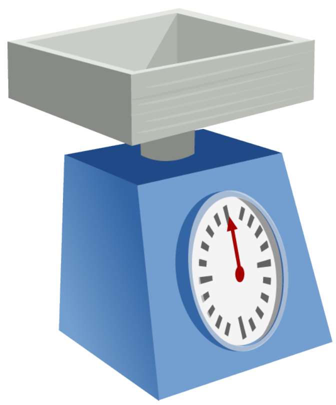
Since the spinner has only 15 sections, we must have $r \leq 15$. Thus, the only solution is $r = 6$. That is, there are 6 red sections.

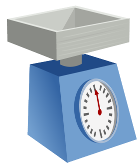
Problem of the Week

Problem E

A Heavy Problem

Four boxes: one blue, one green, one pink, and one yellow, each have a different mass. The mass of each box, in grams, is a positive integer. Inkeri does not know the individual masses of the boxes, but she knows the combined mass of the pink, blue, and green boxes is 13 grams. She also knows the combined mass of the blue, green, and yellow boxes is 17 grams, and the combined mass of the pink, green, and yellow boxes is 19 grams. Determine all possibilities for the individual masses of the boxes.





Problem of the Week

Problem E and Solution

A Heavy Problem

Problem

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Solution

Let b, g, p, and y represent the masses, in grams, of the blue, green, pink, and yellow boxes, respectively. It is given that $b \neq g \neq p \neq y$ and each of the masses is a positive integer. We also know the following:

$$p + b + g = 13 \qquad (1)$$

$$b + g + y = 17 \qquad (2)$$

$$p + g + y = 19 \qquad (3)$$

Subtracting (2) − (1) gives $y - p = 4$, or equivalently, $y = p + 4$.

Subtracting (3) − (2) gives $p - b = 2$, or equivalently, $b = p - 2$.

From here we proceed with two different approaches.

Solution 1

From (1), we obtain $g = 13 - p - b$. Since p, b, g, and y are all different positive integers, we can look at the possible values of p and calculate the values of b, g, and y in each case. Then we can determine if it is a possible set of masses. This is summarized in the following table.

Mass of Pink Box, p	Mass of Blue Box, $b = p - 2$	Mass of Green Box, $g = 13 - p - b$	Mass of Yellow Box, $y = p + 4$	Possible?
1	-1	13	5	No, $b < 1$
2	0	11	6	No, $b < 1$
3	1	9	7	Yes
4	2	7	8	Yes
5	3	5	9	No, $p = g$
6	4	3	10	Yes
7	5	1	11	Yes

We can stop here, because if $p > 7$, then $g < 1$, which is not valid. Therefore, there are four different possibilities for the masses of the blue, green, pink, and yellow boxes, respectively. They are:

$$(b, g, p, y) = (1, 9, 3, 7),\ (2, 7, 4, 8),\ (4, 3, 6, 10),\ (5, 1, 7, 11)$$

Solution 2

This solution is similar to Solution 1. The key difference is that in this solution, we find expressions for b, g, and y, in terms of p. We then determine the smallest and largest possible values for p.

We substitute $y = p + 4$ and $b = p - 2$ into (2) to get an expression for g in terms of p.

$$\begin{aligned} b + g + y &= 17 \\ (p - 2) + g + (p + 4) &= 17 \\ 2p + g + 2 &= 17 \\ g &= 15 - 2p \end{aligned}$$

Since $b = p - 2$ and b is a positive integer, the smallest positive integer value for p will be 3. Otherwise $b < 1$. Since $g = 15 - 2p$ and g is a positive integer, the largest positive integer value for p will be 7. Otherwise $g < 1$. Therefore, the only possible values for p are 3, 4, 5, 6, and 7.

We will now look at each possible value of p, calculate the values of b, g, and y in each case, and determine if it is a possible set of masses.

Mass of Pink Box, p	Mass of Blue Box, $b = p - 2$	Mass of Green Box, $g = 15 - 2p$	Mass of Yellow Box, $y = p + 4$	Possible?
3	1	9	7	Yes
4	2	7	8	Yes
5	3	5	9	No, $p = g$
6	4	3	10	Yes
7	5	1	11	Yes

Therefore, there are four different possibilities for the masses of the blue, green, pink, and yellow boxes, respectively. They are:

$$(b, g, p, y) = (1, 9, 3, 7),\ (2, 7, 4, 8),\ (4, 3, 6, 10),\ (5, 1, 7, 11)$$

Problem of the Week

Problem E

1225 is Even More Special

Did you know that 1225 can be written as the sum of seven consecutive integers?

That is,

$$1225 = 172 + 173 + 174 + 175 + 176 + 177 + 178$$

The notation below illustrates a mathematical short form used for writing the above sum. This notation is called *Sigma Notation*.

$$\sum_{i=172}^{178} i = 1225$$

How many ways can the number 1225 be expressed as the sum of an **even** number of consecutive positive integers?

$$\sum_{i=118}^{127} i = 1225$$

Problem of the Week
Problem E and Solution
1225 is Even More Special

Problem

Did you know that 1225 can be written as the sum of seven consecutive integers?

That is,

$$1225 = 172 + 173 + 174 + 175 + 176 + 177 + 178$$

How many ways can the number 1225 be expressed as the sum of an **even** number of consecutive positive integers?

Solution

Suppose k is even. We can write k consecutive integers as

$$n - \left(\frac{k}{2} - 1\right),\ \ldots,\ n,\ n+1,\ \ldots,\ n + \left(\frac{k}{2} - 1\right),\ n + \frac{k}{2}$$

Here, n and $n+1$ are the middle numbers in the sum, and there are $\frac{k}{2} - 1$ integers less than n in the sum and $\frac{k}{2}$ integers greater than n in the sum.

We can write the sum of these integers in this way:

$$\left(n - \left(\frac{k}{2} - 1\right)\right) + \cdots + n + (n+1) + \cdots + \left(n + \left(\frac{k}{2} - 1\right)\right) + \left(n + \frac{k}{2}\right)$$

This simplifies to $kn + \frac{k}{2}$.

For example, four consecutive integers can be expressed as $n-1$, n, $n+1$, and $n+2$, where n is an integer.
Their sum is $(n-1) + n + (n+1) + (n+2) = 4n + 2$.

Notice that $kn + \frac{k}{2} = k(n + \frac{1}{2})$. Thus, if this sum is equal to 1225, then $k(n + \frac{1}{2}) = 1225$. Multiplying both sides by 2, we have

$$2k\left(n + \frac{1}{2}\right) = 2(1225)$$
$$k(2n+1) = 2450$$

Since n is an integer, then $2n+1$ is an odd integer. Therefore, we're looking for factor pairs of 2450, where one factor is even and the other is odd.

Since $2450 = 2(5^2)(7^2)$, the positive odd divisors of 2450 are 1, 5, 7, 25, 35, 49, 175, 245 and 1225.

For each odd divisor, $2n+1$, of 2450, we determine n and $k = \frac{2450}{2n+1}$. The k integers that sum to 1225 will then be $n - \left(\frac{k}{2} - 1\right), \ldots, n, n+1, \ldots, n + \left(\frac{k}{2} - 1\right), n + \frac{k}{2}$. This is summarized in the table below.

Odd Divisor $(2n+1)$	n	**Number of integers** (k)	**Sum of Integers**
1	0	2450	$(-1224)+(-1223)+\cdots+0+\cdots+1224+1225$
5	2	490	$(-242)+(-241)+\cdots+2+\cdots+246+247$
7	3	350	$(-171)+(-170)+\cdots+3+\cdots+177+178$
25	12	98	$(-36)+(-35)+\cdots+12+\cdots+60+61$
35	17	70	$(-17)+(-16)+\cdots+17+\cdots+51+52$
49	24	50	$0+1+\cdots+24+\cdots+48+49$
175	87	14	$81+82+83+84+85+86+87+88+89+90+91+92+93+94$
245	122	10	$118+119+120+121+122+123+124+125+126+127$
1225	612	2	$612+613$

For $k = 14$, 10, and 2, all integers in the sum are positive.

Thus, there are three ways to express 1225 as the sum of an even number of consecutive positive integers.

Extension: Determine the number of ways the number 1225 can be expressed as the sum of an **odd** number of consecutive positive integers.

Problem of the Week

Problem E

Mystery Function

For some function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are integers, we know the following information:

- the y-intercept is 5,
- $f(2) = -3$,
- $f(4)$ is greater than 40 but less than 50, and
- $f(6)$ is greater than 240 but less than 250.

Determine the value of $f(7)$.

Problem of the Week
Problem E and Solution
Mystery Function

Problem

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- $f(2) = -3$,
- $f(4)$ is greater than 40 but less than 50, and
- $f(6)$ is greater than 240 but less than 250.

Determine the value of $f(7)$.

Solution

Since the y-intercept is 5, it follows that $f(0) = 5$. Thus,

$$\begin{aligned} a(0)^3 + b(0)^2 + c(0) + d &= 5 \\ d &= 5 \end{aligned}$$

We can now write the function as $f(x) = ax^3 + bx^2 + cx + 5$.

Since $f(2) = -3$,

$$\begin{aligned} a(2)^3 + b(2)^2 + c(2) + 5 &= -3 \\ 8a + 4b + 2c + 5 &= -3 \\ 8a + 4b + 2c &= -8 \\ 4a + 2b + c &= -4 \\ c &= -4a - 2b - 4 \qquad (1) \end{aligned}$$

Next we consider $f(4)$.

$$\begin{aligned} f(4) &= a(4)^3 + b(4)^2 + c(4) + 5 \\ &= 64a + 16b + 4c + 5 \\ &= 64a + 16b + 4(-4a - 2b - 4) + 5 \qquad \text{(using equation (1))} \\ &= 64a + 16b - 16a - 8b - 16 + 5 \\ &= 48a + 8b - 11 \end{aligned}$$

Since $f(4) > 40$, it follows that $48a + 8b - 11 > 40$ or $48a + 8b > 51$. Dividing the inequality by 8 gives $6a + b > 6.375$. Similarly, since $f(4) < 50$, it follows that $48a + 8b - 11 < 50$ or $48a + 8b < 61$. Dividing the inequality by 8 gives $6a + b < 7.625$. Since a and b are integers it follows that $6a + b$ is an integer. Thus, since $6a + b > 6.375$ and $6a + b < 7.625$, we can conclude that $6a + b = 7$.

Next we consider $f(6)$.

$$\begin{aligned} f(6) &= a(6)^3 + b(6)^2 + c(6) + 5 \\ &= 216a + 36b + 6c + 5 \\ &= 216a + 36b + 6(-4a - 2b - 4) + 5 \qquad \text{(using equation (1))} \\ &= 216a + 36b - 24a - 12b - 24 + 5 \\ &= 192a + 24b - 19 \end{aligned}$$

Since $f(6) > 240$, it follows that $192a + 24b - 19 > 240$, or $192a + 24b > 259$. Dividing the inequality by 24 gives $8a + b > 10\frac{19}{24}$. Similarly, since $f(6) < 250$, it follows that $192a + 24b - 19 < 250$, or $192a + 24b < 269$. Dividing the inequality by 24 gives $8a + b < 11\frac{5}{24}$. Since a and b are integers it follows that $8a + b$ is an integer. Thus, since $8a + b > 10\frac{19}{24}$ and $8a + b < 11\frac{5}{24}$, we can conclude that $8a + b = 11$.

We now have the following system of equations.

$$6a + b = 7 \qquad (2)$$
$$8a + b = 11 \qquad (3)$$

By subtracting equation (2) from equation (3), we obtain $2a = 4$, or $a = 2$. Substituting $a = 2$ in equation (2) gives $6(2) + b = 7$, and thus $b = -5$.

Substituting $a = 2$ and $b = -5$ in equation (1) gives:

$$\begin{aligned} c &= -4a - 2b - 4 \\ &= -4(2) - 2(-5) - 4 \\ &= -8 + 10 - 4 = -2 \end{aligned}$$

We can now write the function as $f(x) = 2x^3 - 5x^2 - 2x + 5$.

Finally we can determine $f(7)$.

$$\begin{aligned} f(7) &= 2(7)^3 - 5(7)^2 - 2(7) + 5 \\ &= 2(343) - 5(49) - 14 + 5 \\ &= 686 - 245 - 9 = 432 \end{aligned}$$

Therefore, $f(7) = 432$.

NOTE:

We could have written the third bullet point as $40 < f(4) < 50$ and solved the entire inequality at once instead of dealing with the inequality symbols one at a time. While this may be unfamiliar to students, it's a helpful way to solve inequalities. This would have looked as follows.

$$\begin{aligned} 40 < f(4) &< 50 \\ 40 < 48a + 8b - 11 &< 50 \\ 51 < 48a + 8b &< 61 \\ 6.375 < 6a + b &< 7.625 \end{aligned}$$

From here, we can conclude that since $6a + b$ is an integer, then we must have $6a + b = 7$. We could have then used a similar approach to solve the inequalities in $f(6)$.

Computational Thinking (C)

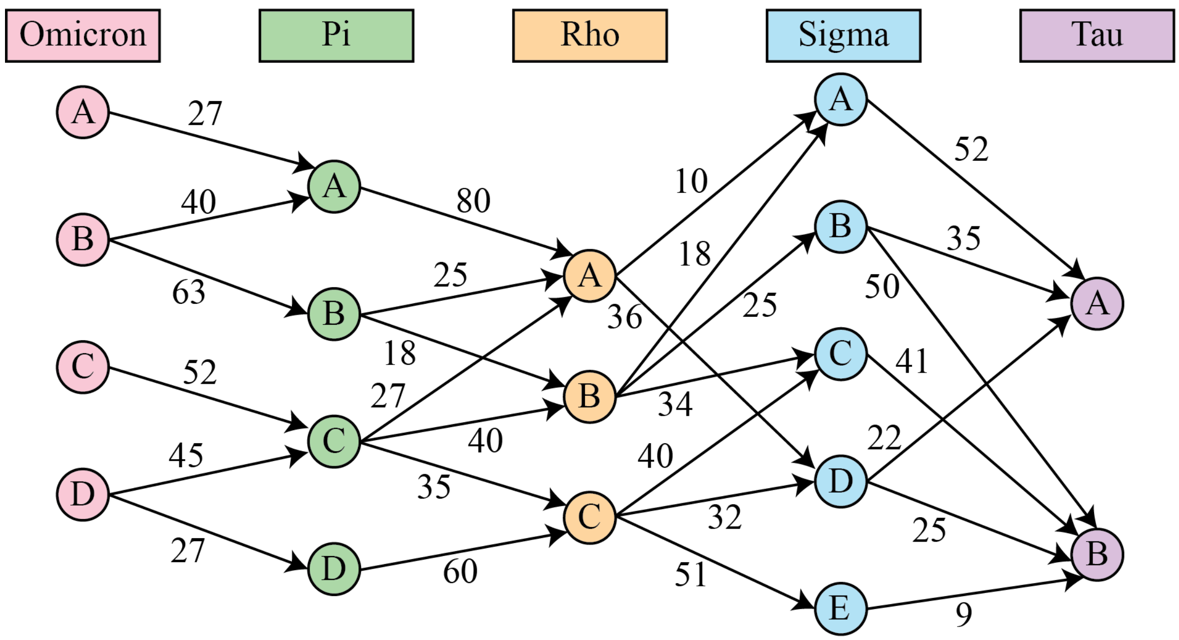
Take me to the cover

Problem of the Week

Problem E

The Fantastic Race

As part of The Fantastic Race, teams need to travel on buses from city to city in the order Omicron to Pi to Rho to Sigma to Tau. Each city has several different bus stations to choose from. Nate has created the following map showing all the different bus routes between the five cities, as well as the travel time, in minutes, for each. The different bus stations within each city are labeled A, B, C, etc.



Which route from Omicron to Tau gives the shortest total travel time?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Problem of the Week

Problem E and Solution

The Fantastic Race

Problem

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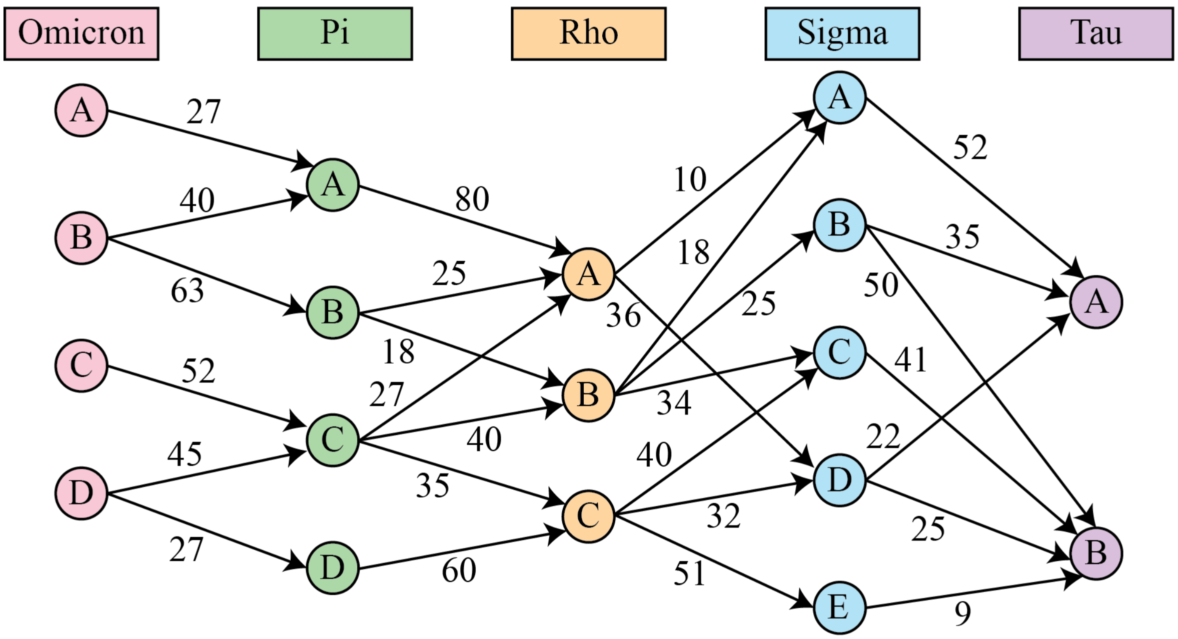


Which route from Omicron to Tau gives the shortest total travel time?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Solution

The shortest total travel time is 130 min, and can be achieved with the following route: Omicron (D) $\rightarrow$ Pi (C) $\rightarrow$ Rho (A) $\rightarrow$ Sigma (D) $\rightarrow$ Tau (A).

In order to find this route, we use a method called *dynamic programming*. In dynamic programming, we systematically build the solution from small pieces to bigger and bigger pieces. This saves us a significant amount of time because we don't have to calculate the total travel time for every possible route.

We start by looking at the routes from Omicron to Pi, and determine the route with the shortest travel time to each of the bus stations in Pi. We will call these the "best" routes. The best route to reach Pi (A) takes 27 min and comes from

Omicron (A). The only route to reach Pi (B) takes 63 min and comes from Omicron (B). The best route to reach Pi (C) takes 45 min and comes from Omicron (D). The only route to reach Pi (D) takes 27 min and comes from Omicron (D). We then ignore all other routes from Omicron to Pi.

Next we look at the routes from Omicron to Rho, using the information we just recorded. The best route to reach Rho (A) takes $45 + 27 = 72$ min and comes from Pi (C). The best route to reach Rho (B) takes $63 + 18 = 81$ min and comes from Pi (B). The best route to reach Rho (C) takes $45 + 35 = 80$ min and comes from Pi (C).

Next we look at the routes from Omicron to Sigma, using the information we just recorded. The best route to reach Sigma (A) takes $72 + 10 = 82$ min and comes from Rho (A). The best route to reach Sigma (B) takes $81 + 25 = 106$ min and comes from Rho (B). The best route to reach Sigma (C) takes $81 + 34 = 115$ min and comes from Rho (B). The best route to reach Sigma (D) takes $72 + 36 = 108$ min and comes from Rho (A). The best route to reach Sigma (E) takes $80 + 51 = 131$ min and comes from Rho (C).

Finally, we look at the routes from Omicron to Tau, using the information we just recorded. The best route to reach Tau (A) takes $108 + 22 = 130$ min and comes from Sigma (D). The best route to reach Tau (B) takes $108 + 25 = 133$ min and comes from Sigma (D).

Thus, the shortest total travel time is 130 min, and is achieved with the route Omicron (D) $\rightarrow$ Pi (C) $\rightarrow$ Rho (A) $\rightarrow$ Sigma (D) $\rightarrow$ Tau (A).

Problem of the Week

Problem E

Three on the ION

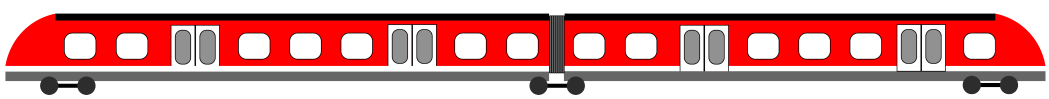
On Wednesday morning, Jayant, Kieron, and Lecia were on the same ION light rail train going to University of Waterloo station. Each person got on at a different station and was eating a different snack on the train. Once they arrived at the university, one person went to physics class, another went to biology class, and the other went to math class. Using the clues below, determine what snack each person was eating, what class they went to, and in which order they got on the train.

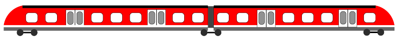
(1) The three people are Kieron, the person eating the bagel, and the person going to math class.

(2) The person eating the apple got on the train before Lecia.

(3) Kieron got on the train 2^{nd} and is *not* going to physics.

(4) The person eating the bagel got on the train before the person eating the granola bar.





Problem of the Week

Problem E and Solution

Three on the ION

Problem

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(3) Kieron got on the train 2^{nd} and is *not* going to physics.

(4) The person eating the bagel got on the train before the person eating the granola bar.

Solution

We can use a table to summarize the information in the clues. If we know that something does not go in a particular cell, we will also include that in the table.

From clue (1), Kieron was not eating a bagel or going to math class. From clue (3), Kieron got on the train 2^{nd} and was not going to physics. We can then conclude that Kieron was going to biology class. This information is summarized in the table below.

	1^{st} to get on	2^{nd} to get on	3^{rd} to get on
Name		Kieron	
Snack		*not* bagel	
Class		biology	

From clue (4), we know that the person eating the bagel was not the 3^{rd} person on the train. Since we know the 2^{nd} person on the train was also not eating a bagel, it follows that the 1^{st} person on the train was eating a bagel.

From clue (2), we know that Lecia was not the 1^{st} person on the train. It follows that she was the 3^{rd} person on the train. Then Jayant was the 1^{st} person on the train.

Also from clues (2) and (4) we can conclude that Kieron was eating the apple and Lecia was eating the granola bar. This information is summarized in the table below.

	1^{st} to get on	2^{nd} to get on	3^{rd} to get on
Name	Jayant	Kieron	Lecia
Snack	bagel	apple	granola bar
Class		biology	

Looking again at clue (1), we can conclude that the person going to math class was the 3^{rd} person on the train. Then the 1^{st} person on the train was going to physics class. The final solution is shown.

	1^{st} to get on	2^{nd} to get on	3^{rd} to get on
Name	Jayant	Kieron	Lecia
Snack	bagel	apple	granola bar
Class	physics	biology	math

Problem of the Week
Problem E
Forgotten Code

Suk-Ja has a combination lock that contains 12 wheels, each with the letters in her name. Thus, the code to open the lock is 12 characters long and contains only the letters S, U, K, J, and A. Unfortunately, Suk-Ja has forgotten the code. She remembers the following information and plans to use this to determine all the possible codes.

- There were exactly three As in the code, located in positions 3, 6, and 12, when numbered from left to right.
- The letter S was always followed by either U or K.
- The letter U was always followed by either K or J.
- The letter K was always followed by either J or A.
- The letter J was always followed by either A or S.
- The letter A was always followed by either S or U.

For example, one possible code is $UJAUKASKJSKA$.

How many different possible codes are there?

Problem of the Week
Problem E and Solution
Forgotten Code

Problem

Suk-Ja has a combination lock that contains 12 wheels, each with the letters in her name. Thus, the code to open the lock is 12 characters long and contains only the letters S, U, K, J, and A. Unfortunately, Suk-Ja has forgotten the code. She remembers the following information and plans to use this to determine all the possible codes.

- There were exactly three As in the code, located in positions 3, 6, and 12, when numbered from left to right.
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- The letter U was always followed by either K or J.
- The letter K was always followed by either J or A.
- The letter J was always followed by either A or S.
- The letter A was always followed by either S or U.

For example, one possible code is $UJAUKASKJSKA$.

How many different possible codes are there?

Solution

To determine the total number of possible codes, we first look at how many ways we can fill positions 1 and 2, then positions 4 and 5, and finally positions 7 to 11.

When filling positions 1 and 2, we note position 3 must be A. Working backwards, we know that position 2 must then be either K or J.

- If position 2 is K, then position 1 can be either S or U.
- If position 2 is J, then position 1 can be either U or K.

Thus, there are 4 ways to fill positions 1 and 2, namely SK, UK, UJ, or KJ.

When filling positions 4 and 5, we note that positions 3 and 6 are both A. Working backwards from position 6, we know that position 5 must then be either K or J. Working forwards from position 3, we know that position 4 must then be either S or U.

- If position 4 is S, then position 5 can be either U or K. However we already determined that position 5 must be either K or J, so position 5 cannot be U.

- If position 4 is U, then position 5 can be either K or J.

Thus, there are 3 ways to fill positions 4 and 5, namely SK, UK, or UJ.

When filling positions 7 to 11, we note that A cannot be in any of these positions, since the only As are in positions 3, 6, and 12. Since A is in position 12, we know that position 11 must be either K or J. Similarly, since A is in position 6, we know that position 7 must then be either S or U.

- If position 7 is S, then position 8 can be either U or K. Thus, positions 7 and 8 can be SU or SK. From here, positions 7, 8, and 9 can be SUK, SUJ, or SKJ. We cannot have SKA since A cannot be in positions 7 to 11. From here, positions 7 to 10 can be $SUKJ$, $SUJS$, or $SKJS$. From here, positions 7 to 11 can be $SUKJS$, $SUJSU$, $SUJSK$, $SKJSU$, or $SKJSK$. However, we already determined that position 11 must be either K or J. Thus, the only two possibilities for positions 7 to 11 with S in position 7 are $SUJSK$ or $SKJSK$.

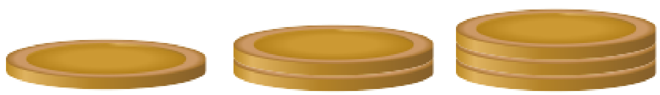
- If position 7 is U, then position 8 can be either K or J. Thus, positions 7 and 8 can be UK or UJ. From here, positions 7, 8, and 9 can be UKJ or UJS. We cannot have UKA or UJA since A cannot be in positions 7 to 11. From here, positions 7 to 10 can be $UKJS$, $UJSU$, or $UJSK$. From here, positions 7 to 11 can be $UKJSU$, $UKJSK$, $UJSUK$, $UJSUJ$, or $UJSKJ$. However, we already determined that position 11 must be either K or J. Thus, the only four possibilities for positions 7 to 11 with U in position 7 are $UKJSK$, $UJSUK$, $UJSUJ$, or $UJSKJ$.

Thus, there are $2 + 4 = 6$ ways to fill positions 7 to 11.

There are 4 ways to fill positions 1 and 2. For each of these, there are 3 ways to fill positions 4 and 5. Thus, there are $4 \times 3 = 12$ ways to fill positions 1, 2, 4, and 5. For each of these, there are 6 ways to fill positions 7 to 11. Thus, there are $12 \times 6 = 72$ different possible codes.

Problem of the Week
Problem E
The Six Dollar Game

A two-player game has players alternating turns removing coins from three piles. At the beginning of the game, the first pile contains 1 coin, the second pile contains 2 coins, and the third pile contains 3 coins. Players take turns removing one or more coins from any **one** pile. On their turn, a player can remove from one coin to all of the coins in a particular pile. The player who removes the last coin wins the game.



Jessie and Jason play this game. Jessie goes first. Jason claims that no matter what Jessie does on his turn, that Jason can win by following his winning strategy.

Describe Jason's winning strategy.

Problem of the Week
Problem E and Solution
The Six Dollar Game

Problem

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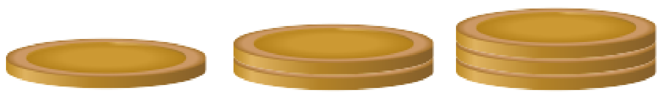


Jessie and Jason play this game. Jessie goes first. Jason claims that no matter what Jessie does on his turn, that Jason can win by following his winning strategy.

Describe Jason's winning strategy.

Solution

There are six possibilities for Jessie's first move: he can remove 1 coin from the first, second, or third pile, or 2 coins from the second or third pile, or all 3 coins from the third pile. In each case, Jason can always remove coins in some way so that when his first turn is over there are two piles with the same number of coins and one pile with no coins remaining, as follows.

- **Case 1:** Jessie removes 1 coin from the first pile.

  Jason should remove 1 coin from the third pile on his first turn. The result will be 0 coins in the first pile, 2 coins in the second pile, and 2 coins in the third pile.

- **Case 2:** Jessie removes 1 coin from the second pile.

  Jason should remove all 3 coins from the third pile on his first turn. The result will be 1 coin in the first pile, 1 coin in the second pile, and 0 coins in the third pile.

- **Case 3:** Jessie removes 1 coin from the third pile.

  Jason should remove 1 coin from the first pile on his first turn. The result will be 0 coins in the first pile, 2 coins in the second pile, and 2 coins in the third pile.

- **Case 4:** Jessie removes 2 coins from the second pile.

  Jason should remove 2 coins from the third pile on his first turn. The result will be 1 coin in the first pile, 0 coins in the second pile, and 1 coin in the third pile.

- **Case 5:** Jessie removes 2 coins from the third pile.

  Jason should remove 2 coins from the second pile on his first turn. The result will be 1 coin in the first pile, 0 coins in the second pile, and 1 coin in the third pile.

- **Case 6:** Jessie removes 3 coins from the third pile.

  Jason should remove 1 coin from the second pile on his first turn. The result will be 1 coin in the first pile, 1 coin in the second pile, and 0 coins in the third pile.

In each of the six cases, Jason leaves Jessie with only two piles, each containing the same number of coins. More specifically, after Jason's first turn there will be either two piles with 2 coins in each or two piles with 1 coin in each.

If the two remaining piles each contain 1 coin, Jessie must then remove 1 coin from one of the piles on his second turn. Then on his second turn Jason removes the final coin and wins.

If the two remaining piles each contain 2 coins, Jessie has two choices for his second turn: Jessie can either remove 2 coins from one of the piles or 1 coin from one of the piles.

- If Jessie removes 2 coins from one of the piles, Jason then removes 2 coins from the remaining pile and wins.
- If Jessie removes 1 coin from one of the piles on his second turn, then Jason responds by removing 1 coin from the other pile. After Jason's second turn there would be two piles remaining, each containing 1 coin. Jessie must then remove 1 coin from one of the piles on his third turn. Jason then removes the final coin on his third turn and wins.

In summary, Jason's strategy is as follows. After Jessie's first turn, Jason removes coin(s) from a pile so that only two piles remain and each pile contains the same number of coins. On his second and possible third turn, he copies what Jessie did on his previous turn, but to the pile he did not touch.

For Further Thought:

Jessie proposes two new versions of the coin game. For these new versions, does either player have a winning strategy?

(a) Suppose there are just two piles to start, one pile has 13 coins and the other has 17 coins. On any turn a player can remove any number of coins from one of the piles. Players alternate turns and the player to remove the last coin wins.

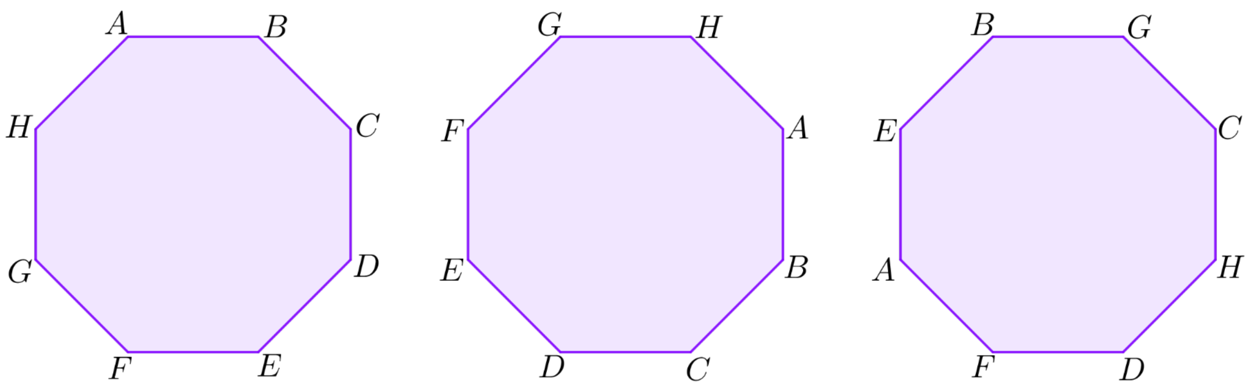
(b) Suppose there are three piles of coins to start, one pile has 2 coins, a second pile has 4 coins, and the third pile has 5 coins. On any turn a player can remove any number of coins from one of the piles. Players alternate turns and the player to remove the last coin wins.

Data Management (D)

Take me to the cover

Problem of the Week
Problem E
Octosquares?

The eight vertices of a regular octagon are randomly labelled A, B, C, D, E, F, G, and H. Each letter is used exactly once. For example, the images below show three different ways to label the vertices of the octagon.

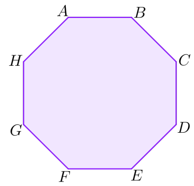


It can be shown that the only way to form a square whose vertices are also vertices of the octagon is by drawing a line segment between every other vertex in the regular octagon.

For example, in the first labelling example, both $ACEG$ and $BDFH$ are squares. These are the only two squares that can be formed using that specific labelling of the octagon. Similarly, in the second example, $ACEG$ and $BDFH$ are the only two squares that can be formed, and in the third example, $ABCD$ and $EGHF$ are the only two squares that can be formed.

If the vertices of a regular octagon are randomly labelled A, B, C, D, E, F, G, and H and each letter is used exactly once, what is the probability that $ABCD$ is a square?

That is, what is the probability that the shape formed by connecting the vertex labelled A to the vertex labelled B, the vertex B to the vertex labelled C, the vertex labelled C to the vertex labelled D, and the vertex labelled D to the vertex labelled A, is a square?



Problem of the Week
Problem E and Solution
Octosquares?

Problem

If the vertices of a regular octagon are randomly labelled A, B, C, D, E, F, G, and H and each letter is used exactly once, what is the probability that $ABCD$ is a square?

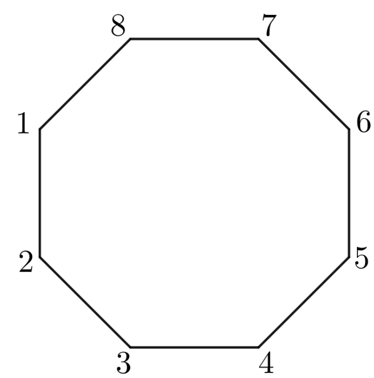
Solution

Solution 1

To determine the probability, we determine the number of ways to label the vertices of the regular octagon so that $ABCD$ forms a square and divide by the total number of ways the regular octagon can be labelled.

First, we determine the total number of ways that the vertices of a regular octagon can be labelled A, B, C, D, E, F, G, and H.

Let's start with the top left vertex. There are 8 possible ways to label it (it can be labelled as A, B, C, D, E, F, G, or H). Moving clockwise, the next vertex can be labelled 7 different ways (it can be assigned any letter other than the letter assigned to the previous vertex). Moving clockwise, the next vertex can be labelled 6 different ways (it can be assigned any letter other than the 2 that have already been used). Moving clockwise, the next vertex can be assigned 5 different ways, and so on. Once we reach the last vertex there will only be 1 letter left, so it can be assigned a letter only 1 way.



Therefore, there are

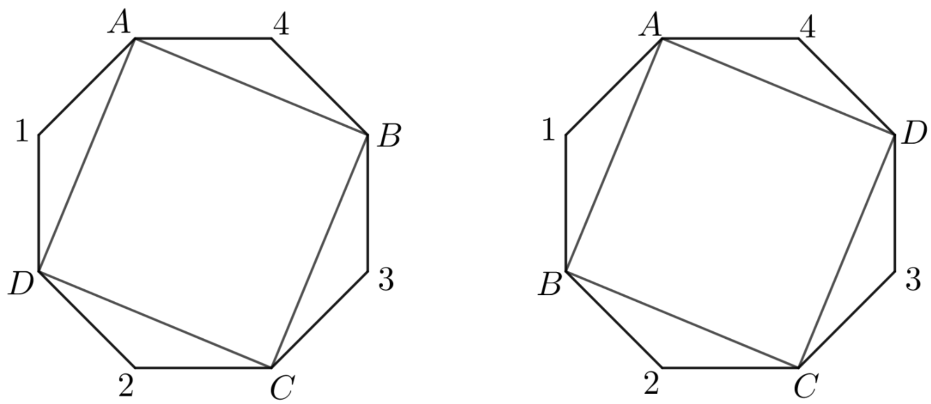
$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = 40\,320$$

different ways to label the regular octagon with the letters A, B, C, D, E, F, G, and H.

Now, let's determine how many of the 40 320 labellings result in $ABCD$ forming a square.

Let's suppose vertex A is the top left vertex. Then there are two possible ways to label B, C, and D so that $ABCD$ forms a square.

For each of these two labellings, there are 4 choices for labelling the vertex to the right of A (it can be assigned E, F, G, or H). Given the labelling of that vertex, moving clockwise, there are 3 choices for the next unlabelled vertex, then 2 choices for the next unlabelled vertex, and then 1 choice for the last unlabelled vertex.



Therefore, for each case, there are $4 \times 3 \times 2 \times 1 = 24$ ways to label the remaining vertices. Therefore, there are $24 + 24 = 48$ ways to label the regular octagon with A being the top left vertex and $ABCD$ forming a square.

Using a similar argument, we can see that for any vertex that A can be assigned to, there will be 48 ways to label the regular octagon so that $ABCD$ forms a square. Since A can be assigned to 8 different vertices, there are $8 \times 48 = 384$ different ways to label the regular octagon so that $ABCD$ forms a square.

Therefore, the probability that $ABCD$ forms a square is $\dfrac{384}{40\,320} = \dfrac{1}{105}$.

Solution 2

The first solution counts the number of arrangements where $ABCD$ forms a square, and divides by the total number of possible arrangements. This solution uses a more direct probability argument.

The label A can go anywhere.

There is now two spots where label B can be placed to form a square, so a $\frac{2}{7}$ chance that the B will be placed in a location to form a square.

There is now one spot where C must be placed (across from the A), so a $\frac{1}{6}$ chance that the C will be placed in a location to form a square.

Since there are 5 vertices left to be labelled, there is now a $\frac{1}{5}$ chance that the D will be placed in the only valid location to form a square.

Thus, the probability that $ABCD$ forms a square is

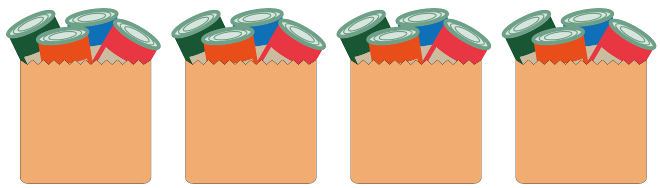
$$\frac{2}{7} \times \frac{1}{6} \times \frac{1}{5} = \frac{2}{210} = \frac{1}{105}$$

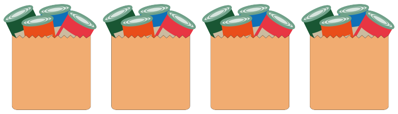
Problem of the Week

Problem E

Free Time

Pearl coordinates volunteers to help collect and sort donations at the food bank on Mondays, Wednesdays, and Fridays. Of her volunteers, 50% are available on Mondays, 80% are available on Wednesdays, and 90% are available on Fridays. A total of 18 volunteers are available on all three days, and all other volunteers are available on exactly two of the three days. How many volunteers are there in total?





Problem of the Week

Problem E and Solution

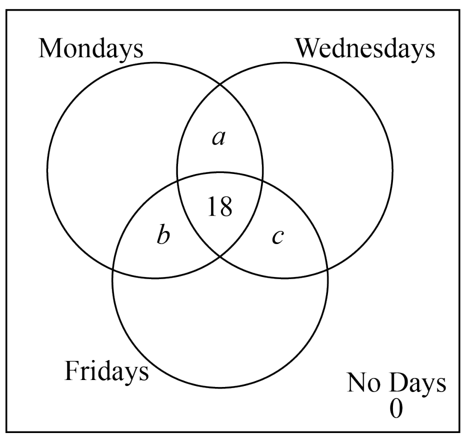
Free Time

Problem

Pearl coordinates volunteers to help collect and sort donations at the food bank on Mondays, Wednesdays, and Fridays. Of her volunteers, 50% are available on Mondays, 80% are available on Wednesdays, and 90% are available on Fridays. A total of 18 volunteers are available on all three days, and all other volunteers are available on exactly two of the three days. How many volunteers are there in total?

Solution

Let a be the number of volunteers who are available on Mondays and Wednesdays, but not Fridays. Let b be the number of volunteers who are available on Mondays and Fridays, but not Wednesdays. Let c be the number of volunteers who are available on Wednesdays and Fridays, but not Mondays. We note that none of the volunteers are available on zero or only one of the three days, and that 18 volunteers are available on all three days. We summarize this information in the following Venn diagram.



Let n be the total number of volunteers. Then, $n = a + b + c + 18$. From the given information,

- 50% of the volunteers are available on Mondays, so $0.5n = a + b + 18$.
- 80% of the volunteers are available on Wednesdays, so $0.8n = a + c + 18$.
- 90% of the volunteers are available on Fridays, so $0.9n = b + c + 18$.

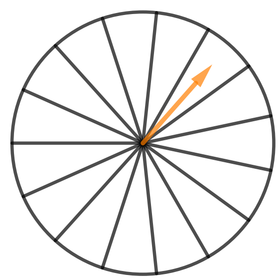
Since $n = a + b + c + 18$, it follows that $2n = 2a + 2b + 2c + 36$. Then,

$$\begin{aligned} 2n &= 2a + 2b + 2c + 36 \\ &= (a + b + 18) + (a + c + 18) + (b + c) \\ &= 0.5n + 0.8n + 0.9n - 18 \\ &= 2.2n - 18 \\ 18 &= 0.2n \\ n &= 90 \end{aligned}$$

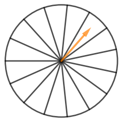
Thus, there are 90 volunteers in total.

Problem of the Week
Problem E
Spinning Red

A spinner is divided into 15 equal sections. Each section is coloured either red or yellow. An arrow is attached to the centre of the spinner. Jamal spins the arrow two times. If there is a 64% chance of landing on red in at least one of the two spins, how many red sections are there?



NOTE: You may use the following fact from probability theory: If the probability of event A occurring is a, the probability of event B occurring is b, and the events are not dependent on each other, then the probability of both events occurring is $a \times b$.



Problem of the Week
Problem E and Solution
Spinning Red

Problem

A spinner is divided into 15 equal sections. Each section is coloured either red or yellow. An arrow is attached to the centre of the spinner. Jamal spins the arrow two times. If there is a 64% chance of landing on red in at least one of the two spins, how many red sections are there?

NOTE: You may use the following fact from probability theory: If the probability of event A occurring is a, the probability of event B occurring is b, and the events are not dependent on each other, then the probability of both events occurring is $a \times b$.

Solution
Solution 1

Let n be the number of yellow sections. Therefore, for each spin, the probability the spinner lands on a yellow section (and therefore does not land on a red section) is $\frac{n}{15}$. Since the result of each spin does not depend on the previous spin,

$$\begin{aligned} P(\text{not red on either spin}) &= P(\text{not red on first spin}) \times P(\text{not red on second spin}) \\ &= \frac{n}{15} \times \frac{n}{15} \\ &= \left(\frac{n}{15}\right)^2 \end{aligned}$$

The probability of landing on at least one red in two spins is 0.64. So, the probability of not landing on red in either of the two spins is $1 - 0.64 = 0.36$. That is,

$$\left(\frac{n}{15}\right)^2 = 0.36$$

Since $n \geq 0$, this simplifies to $\frac{n}{15} = 0.6$, and so $n = 9$.

Since n is the number of yellow sections, there are $15 - 9 = 6$ red sections.

Solution 2

Let r be the number of red sections. Therefore, the number of yellow sections is $15 - r$. Also, for each spin, the probability the spinner lands on a red section is $\frac{r}{15}$ and the probability the spinner lands on a yellow section is $\frac{15-r}{15}$.

If Jamal lands on red in at least one spin, then he may land red on the first spin only, red on the second spin only, or red on both spins.

If Jamal lands on red on his first spin only, then his second spin must land on yellow. Since the results of each spin do not depend on each other, the probability that he spins red on his first spin and yellow on his second spin is $\frac{r}{15} \times \frac{15-r}{15} = \frac{r(15-r)}{15^2}$.

If Jamal lands on red on his second spin only, then his first spin must land on yellow. Thus, the probability Jamal spins yellow on his first spin and red on his second spin is $\frac{15-r}{15} \times \frac{r}{15} = \frac{r(15-r)}{15^2}$.

The probability that Jamal lands on red on his first spin and again on his second spin is $\frac{r}{15} \times \frac{r}{15} = \frac{r^2}{15^2}$.

The probability of getting at least one red in the two spins is equal to the probability that he lands red on his first spin only, plus the probability that he lands red on his second spin only, plus the probability that he lands red on both spins. That is,

$$\frac{r(15-r)}{15^2} + \frac{r(15-r)}{15^2} + \frac{r^2}{15^2} = 0.64$$

Multiplying both sides by 15^2 gives

$$r(15-r) + r(15-r) + r^2 = 144$$

This simplifies to $r^2 - 30r + 144 = 0$. By factoring, we obtain $(r-6)(r-24) = 0$. Therefore, $r = 6$ or $r = 24$.

Since the spinner has only 15 sections, we must have $r \leq 15$. Thus, the only solution is $r = 6$. That is, there are 6 red sections.

Solution 3

Let r be the number of red sections. Therefore, the number of yellow sections is $15 - r$. Also, for each spin, the probability the spinner lands on a red section is $\frac{r}{15}$ and the probability the spinner lands on a yellow section is $\frac{15-r}{15}$.

If Jamal lands on red in at least one spin, then the first red occurs on his first spin or on his second spin.

If Jamal lands on the first red on his first spin, then on his first spin he spins a red, and on his second spin he spins any colour. Since the results of each spin do not depend on each other, the probability that the first red occurs on his first spin is $\frac{r}{15} \times \frac{15}{15} = \frac{r}{15}$.

If Jamal lands on the first red on the second spin, then on his first spin he spins a yellow, and on his second spin he spins a red. The probability of this is $\frac{15-r}{15} \times \frac{r}{15} = \frac{(15-r)r}{15^2}$.

The probability of spinning at least one red on the two spins is equal to the probability that he lands the first red on his first spin, plus the probability that he lands the first red on his second spin. That is,

$$\begin{aligned} \frac{r}{15} + \frac{(15-r)r}{15^2} &= 0.64 \\ 15r + (15-r)r &= 144 \\ r^2 - 30r + 144 &= 0 \end{aligned}$$

Using the quadratic formula, we find $r = \dfrac{30 \pm \sqrt{30^2 - 4(1)(144)}}{2} = \dfrac{30 \pm 18}{2} = 24, 6$.

Since the spinner has only 15 sections, we must have $r \leq 15$. Thus, the only solution is $r = 6$. That is, there are 6 red sections.

Geometry & Measurement (G)

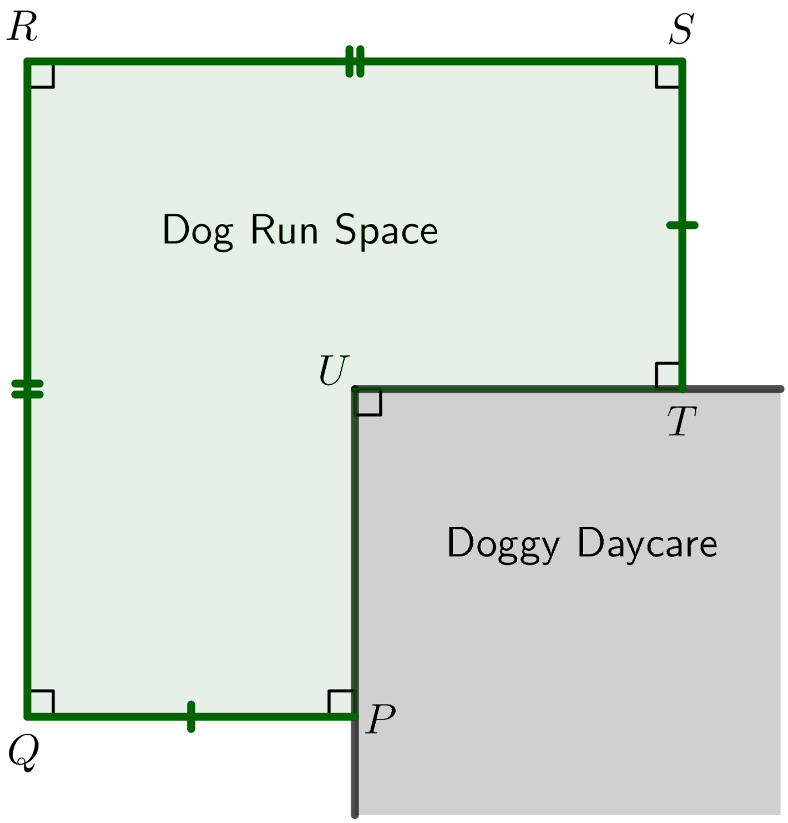
Take me to the cover

Problem of the Week

Problem E

Run Dog Run

At POTW Doggy Care, there is a need for a new outdoor dog run space. The layout of the dog run space is represented by $PQRSTU$ in the diagram below.



The lengths of the two longer sides, QR and RS, are to be the same, and the lengths of the two shorter sides, PQ and ST, are to be the same. There will be right angles at each corner.

The dog run space is to be built using a fence along PQ, QR, RS, and ST, and using the walls of the daycare along PU and TU. The total fencing to be used is 30 m. Determine the dimensions of the dog run space that will give the maximum area for the dog run.

Problem of the Week
Problem E and Solution
Run Dog Run

Problem

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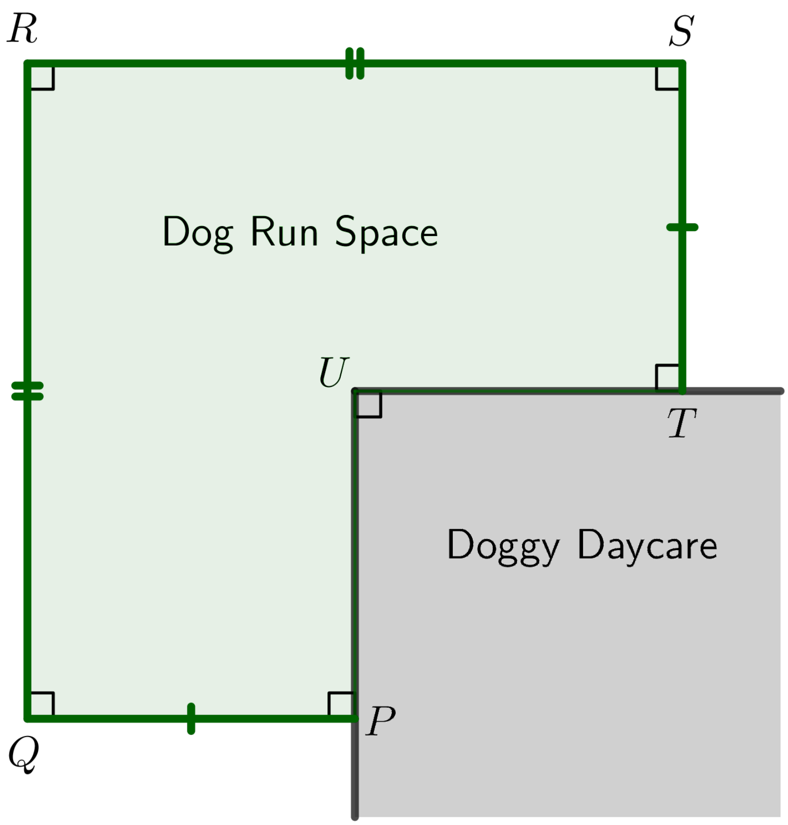

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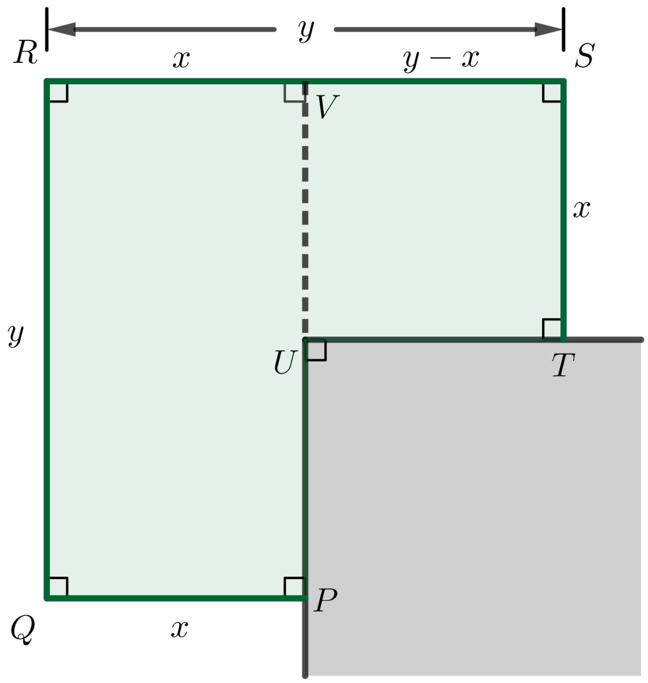
The dog run space is to be built using a fence along PQ, QR, RS, and ST, and using the walls of the daycare along PU and TU. The total fencing to be used is 30 m. Determine the dimensions of the dog run space that will give the maximum area for the dog run.

Solution

Extend PU to RS, letting the intersection point be V. Then $PV \perp RS$.

Let x represent the lengths, in metres, of both PQ and ST. Let y represent the lengths, in metres, of both QR and RS. Since $PQRV$ is a rectangle, $RV = PQ = x$ and $VS = RS - RV = y - x$.



The total length of fencing from P to Q to R to S to T is

$$PQ + QR + RS + ST = x + y + y + x = 2x + 2y$$

Since the total amount of fencing used is 30 m, we have $2x + 2y = 30$. Thus, $x + y = 15$ and $y = 15 - x$.

$$\begin{aligned}\text{Area of dog run} &= \text{Area } PQRV + \text{Area } VSTU \\ &= QR \times RV + VS \times ST \\ &= yx + (y - x)x \\ &= 2xy - x^2\end{aligned}$$

Substituting $y = 15 - x$, this becomes

$$\begin{aligned}\text{Area of dog run} &= 2x(15 - x) - x^2 \\ &= 30x - 2x^2 - x^2 \\ &= -3x^2 + 30x\end{aligned}$$

Completing the square, we have

$$\begin{aligned}\text{Area of dog run} &= -3(x^2 - 10x) \\ &= -3(x^2 - 10x + 5^2 - 5^2) \\ &= -3(x^2 - 10x + 25) + 75 \\ &= -3(x - 5)^2 + 75\end{aligned}$$

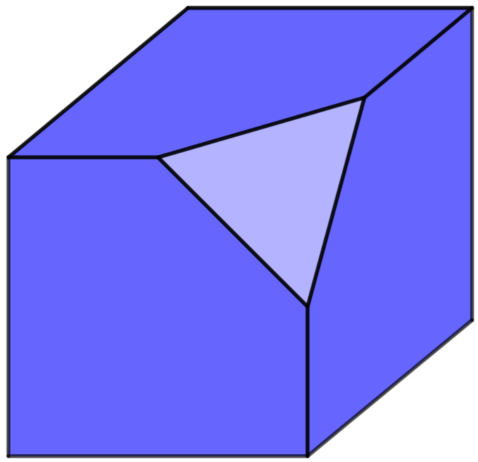
This is the equation of a parabola which opens down from a vertex of $(5, 75)$. Thus, the maximum area is 75 m^2, and occurs when $x = 5$ m. When $x = 5$, we have $y = 15 - x = 15 - 5 = 10$ m.

Therefore, if $QR = RS = 10$ m and $PQ = ST = 5$ m, this gives a maximum area of 75 m^2.

Problem of the Week
Problem E
Cutting Corners

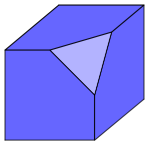
Edwin has a cube with edge length 8 cm. He cuts off a corner by doing the following steps:

1. He finds the midpoint of each edge.
2. He then makes a cut through three of these points on adjacent edges.



He then removes the other seven corners by making similar cuts.

Edwin thinks that the new shape will have a smaller total surface area than the original cube. Show that Edwin is right by finding how much less the new surface area is.



Problem of the Week

Problem E and Solution

Cutting Corners

Problem

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1. He finds the midpoint of each edge.
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He then removes the other seven corners by making similar cuts.

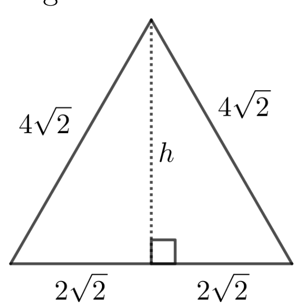
Edwin thinks that the new shape will have a smaller total surface area than the original cube. Show that Edwin is right by finding how much less the new surface area is.

Solution

We will consider one corner, determine the surface area decrease there, and then multiply the result by 8 to account for the eight corners. At each corner, since the cut is made through the midpoints of three adjacent edges of the cube, the surface areas of three identical isosceles right-angled triangles, each with 4 cm base and 4 cm height, are removed and replaced by the surface area of a single equilateral triangle.

Since the isosceles right-angled triangles each have 4 cm base and 4 cm height, each has area equal to $\frac{1}{2}(4)(4) = 8$ cm^2.

Each side length of the equilateral triangle is formed by the hypotenuse of one of the isosceles right-angled triangles. Using the Pythagorean Theorem, we calculate the length of the hypotenuse of each right-angled triangle to be $\sqrt{4^2+4^2} = \sqrt{32} = 4\sqrt{2}$ cm. Thus, the remaining equilateral triangle has sides of length $4\sqrt{2}$ cm. Let h be the height of the equilateral triangle. Since the triangle is equilateral, the height bisects the base.



By the Pythagorean Theorem, $h^2 = (4\sqrt{2})^2 - (2\sqrt{2})^2 = 32 - 8 = 24$. Since $h > 0$, we have $h = \sqrt{24} = 2\sqrt{6}$ cm.

Therefore, the area of the remaining equilateral triangle is $\frac{1}{2}(4\sqrt{2})(2\sqrt{6}) = 4\sqrt{12} = 8\sqrt{3}$ cm^2.

At each corner, the surface area is increased by the area of the equilateral triangle and decreased by the areas of the three right-angled triangles. Therefore, removing a corner changes the surface area by $8\sqrt{3} - 3(8) = (8\sqrt{3} - 24)$ cm^2. Since $8\sqrt{3} < 24$, this result is negative and the surface area is decreased in each corner. Therefore, removing a corner decreases the surface area by $(24 - 8\sqrt{3})$ cm^2.

Since there are eight corners, the total decrease in surface area is

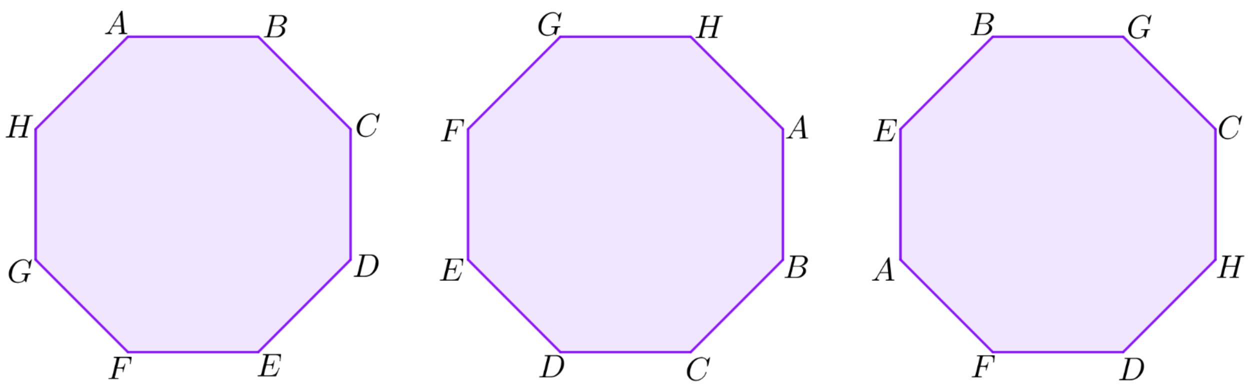
$$8 \times (24 - 8\sqrt{3}) = 192 - 64\sqrt{3} \doteq 81.1 \text{ cm}^2$$

Problem of the Week

Problem E

Octosquares?

The eight vertices of a regular octagon are randomly labelled A, B, C, D, E, F, G, and H. Each letter is used exactly once. For example, the images below show three different ways to label the vertices of the octagon.

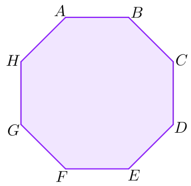


It can be shown that the only way to form a square whose vertices are also vertices of the octagon is by drawing a line segment between every other vertex in the regular octagon.

For example, in the first labelling example, both $ACEG$ and $BDFH$ are squares. These are the only two squares that can be formed using that specific labelling of the octagon. Similarly, in the second example, $ACEG$ and $BDFH$ are the only two squares that can be formed, and in the third example, $ABCD$ and $EGHF$ are the only two squares that can be formed.

If the vertices of a regular octagon are randomly labelled A, B, C, D, E, F, G, and H and each letter is used exactly once, what is the probability that $ABCD$ is a square?

That is, what is the probability that the shape formed by connecting the vertex labelled A to the vertex labelled B, the vertex B to the vertex labelled C, the vertex labelled C to the vertex labelled D, and the vertex labelled D to the vertex labelled A, is a square?



Problem of the Week
Problem E and Solution
Octosquares?

Problem

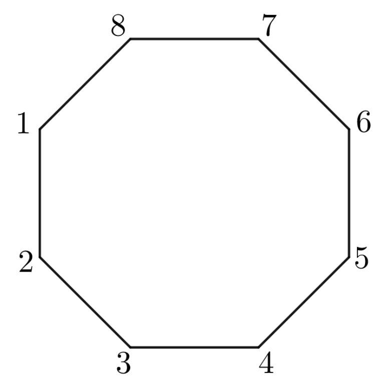
If the vertices of a regular octagon are randomly labelled A, B, C, D, E, F, G, and H and each letter is used exactly once, what is the probability that $ABCD$ is a square?

Solution

Solution 1

To determine the probability, we determine the number of ways to label the vertices of the regular octagon so that $ABCD$ forms a square and divide by the total number of ways the regular octagon can be labelled.

First, we determine the total number of ways that the vertices of a regular octagon can be labelled A, B, C, D, E, F, G, and H.



Let's start with the top left vertex. There are 8 possible ways to label it (it can be labelled as A, B, C, D, E, F, G, or H). Moving clockwise, the next vertex can be labelled 7 different ways (it can be assigned any letter other than the letter assigned to the previous vertex). Moving clockwise, the next vertex can be labelled 6 different ways (it can be assigned any letter other than the 2 that have already been used). Moving clockwise, the next vertex can be assigned 5 different ways, and so on. Once we reach the last vertex there will only be 1 letter left, so it can be assigned a letter only 1 way.

Therefore, there are

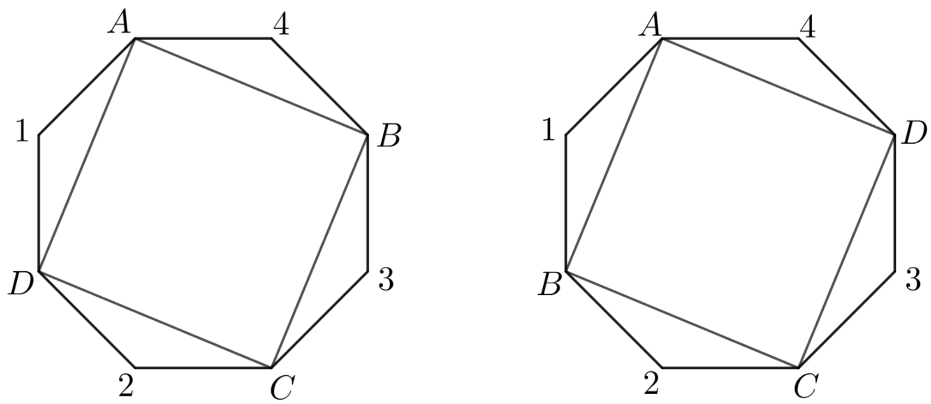
$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = 40\,320$$

different ways to label the regular octagon with the letters A, B, C, D, E, F, G, and H.

Now, let's determine how many of the $40\,320$ labellings result in $ABCD$ forming a square.

Let's suppose vertex A is the top left vertex. Then there are two possible ways to label B, C, and D so that $ABCD$ forms a square.

For each of these two labellings, there are 4 choices for labelling the vertex to the right of A (it can be assigned E, F, G, or H). Given the labelling of that vertex, moving clockwise, there are 3 choices for the next unlabelled vertex, then 2 choices for the next unlabelled vertex, and then 1 choice for the last unlabelled vertex.



Therefore, for each case, there are $4 \times 3 \times 2 \times 1 = 24$ ways to label the remaining vertices. Therefore, there are $24 + 24 = 48$ ways to label the regular octagon with A being the top left vertex and $ABCD$ forming a square.

Using a similar argument, we can see that for any vertex that A can be assigned to, there will be 48 ways to label the regular octagon so that $ABCD$ forms a square. Since A can be assigned to 8 different vertices, there are $8 \times 48 = 384$ different ways to label the regular octagon so that $ABCD$ forms a square.

Therefore, the probability that $ABCD$ forms a square is $\dfrac{384}{40\,320} = \dfrac{1}{105}$.

Solution 2

The first solution counts the number of arrangements where $ABCD$ forms a square, and divides by the total number of possible arrangements. This solution uses a more direct probability argument.

The label A can go anywhere.

There is now two spots where label B can be placed to form a square, so a $\frac{2}{7}$ chance that the B will be placed in a location to form a square.

There is now one spot where C must be placed (across from the A), so a $\frac{1}{6}$ chance that the C will be placed in a location to form a square.

Since there are 5 vertices left to be labelled, there is now a $\frac{1}{5}$ chance that the D will be placed in the only valid location to form a square.

Thus, the probability that $ABCD$ forms a square is

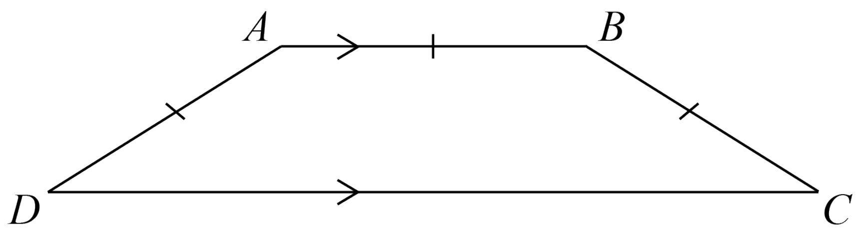
$$\frac{2}{7} \times \frac{1}{6} \times \frac{1}{5} = \frac{2}{210} = \frac{1}{105}$$

Problem of the Week

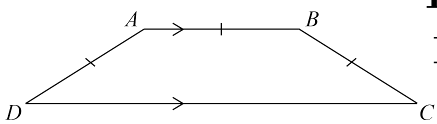
Problem E

Trapped

In trapezoid $ABCD$, sides AB and CD are parallel, and the lengths of sides AD, AB, and BC are equal.



If the perpendicular distance between AB and CD is 8 units, and the length of CD is equal to 4 more than half the sum of the other three side lengths, determine the area and perimeter of trapezoid $ABCD$.



Problem of the Week
Problem E and Solution
Trapped

Problem

In trapezoid $ABCD$, sides AB and CD are parallel, and the lengths of sides AD, AB, and BC are equal. If the perpendicular distance between AB and CD is 8 units, and the length of CD is equal to 4 more than half the sum of the other three side lengths, determine the area and perimeter of trapezoid $ABCD$.

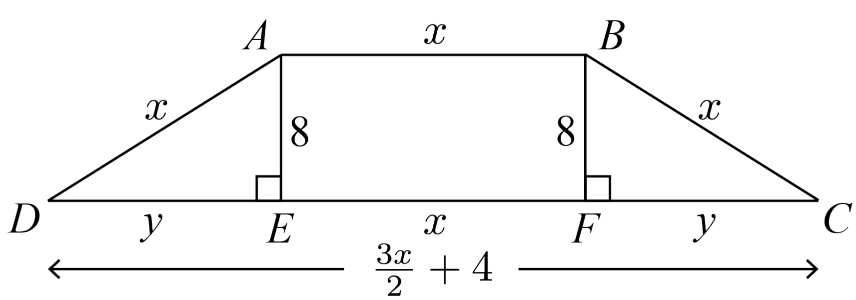
Solution

Let x represent the length of AD. Then $AD = AB = BC = x$. Since CD is equal to 4 more than half the sum of the other three side lengths, $CD = \frac{3x}{2} + 4$. Note that since $x > 0$, $\frac{3}{2}x + 4 > x$ and so CD is longer than AB.

Draw altitudes from A and B meeting CD at E and F, respectively. Then $AE = BF = 8$.

Let y represent the length of DE. We can show that $DE = CF$ using the Pythagorean Theorem as follows: $DE^2 = AD^2 - AE^2 = x^2 - 8^2 = x^2 - 64$ and $CF^2 = BC^2 - BF^2 = x^2 - 8^2 = x^2 - 64$. Then $CF^2 = x^2 - 64 = DE^2$, so $CF = DE = y$ since $CF > 0$.

Since $\angle AEF = \angle BFE = 90^\circ$ and AB is parallel to CD, it follows that $\angle BAE = \angle ABF = 90^\circ$ and $ABFE$ is a rectangle so $AB = EF = x$.



We can now determine the relationship between x and y.

$$\begin{aligned} CD &= DE + EF + CF \\ \frac{3x}{2} + 4 &= y + x + y \\ \frac{x}{2} + 4 &= 2y \\ \frac{x}{4} + 2 &= y \end{aligned}$$

So $DE = CF = \frac{x}{4} + 2$. We then use the Pythagorean Theorem in $\triangle AED$.

$$\begin{aligned} AD^2 &= AE^2 + DE^2 \\ x^2 &= 8^2 + \left(\frac{x}{4} + 2\right)^2 \\ x^2 &= 64 + \frac{x^2}{16} + x + 4 \\ 0 &= \frac{15x^2}{16} - x - 68 \end{aligned}$$

Using the quadratic formula, we can determine the value of x.

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1 - 4\left(\frac{15}{16}\right)(-68)}}{2\left(\frac{15}{16}\right)} \\ &= \frac{1 \pm \sqrt{256}}{\frac{15}{8}} \\ &= \frac{8 \pm 128}{15} \end{aligned}$$

Thus, $x = -8$ or $x = \frac{136}{15}$. Since $x > 0$, we can conclude that $x = \frac{136}{15}$, so $AD = AB = BC = \frac{136}{15}$. Then $CD = \left(\frac{3}{2}\right)\left(\frac{136}{15}\right) + 4 = \frac{88}{5}$. Then we can calculate the area and perimeter of trapezoid $ABCD$.

$$\begin{aligned} \text{Area of } ABCD &= \frac{1}{2} \times AE \times (AB + CD) \\ &= \frac{1}{2} \times 8 \times \left(\frac{88}{5} + \frac{136}{15}\right) \\ &= \frac{320}{3} \end{aligned}$$

$$\begin{aligned} \text{Perimeter of } ABCD &= DA + AB + BC + CD \\ &= 3\left(\frac{136}{15}\right) + \frac{88}{5} \\ &= \frac{224}{5} \end{aligned}$$

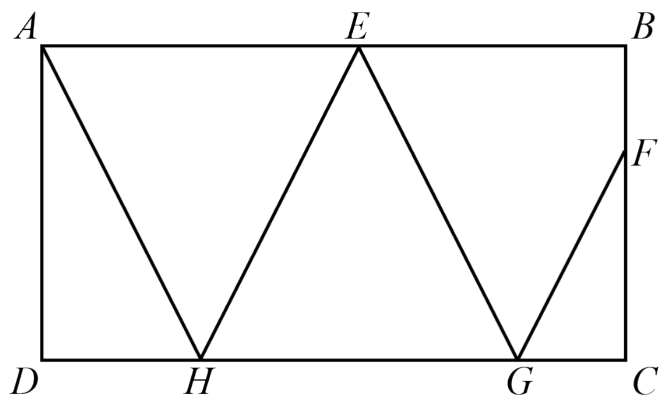
Therefore the area of trapezoid $ABCD$ is $\frac{320}{3}$ units^2 and the perimeter is $\frac{224}{5}$ units.

Problem of the Week

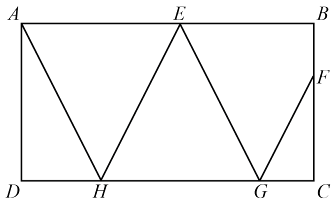
Problem E

Zigzagged

A fence is to be constructed in a zigzag pattern inside a rectangular field, as shown.



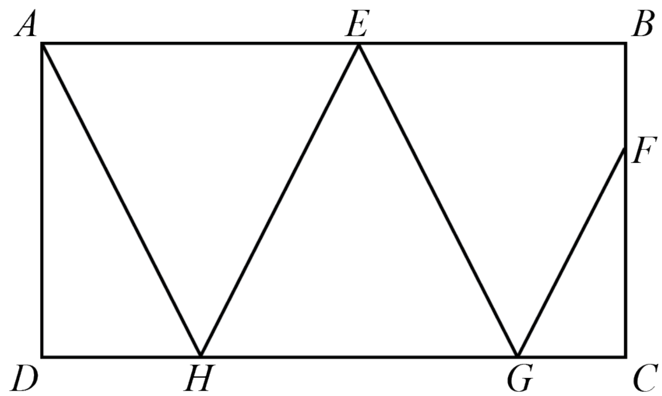
The fence will be constructed so that $\angle AHD = \angle EHG$, $\angle AEH = \angle BEG$, $\angle EGH = \angle FGC$, and $CF = 12$ m. If $AB = 36$ m and $AD = 20$ m, determine the total length of fencing required. That is, determine the value of $AH + EH + EG + FG$.



Problem of the Week
Problem E and Solution
Zigzagged

Problem

A fence is to be constructed in a zigzag pattern inside a rectangular field, as shown.

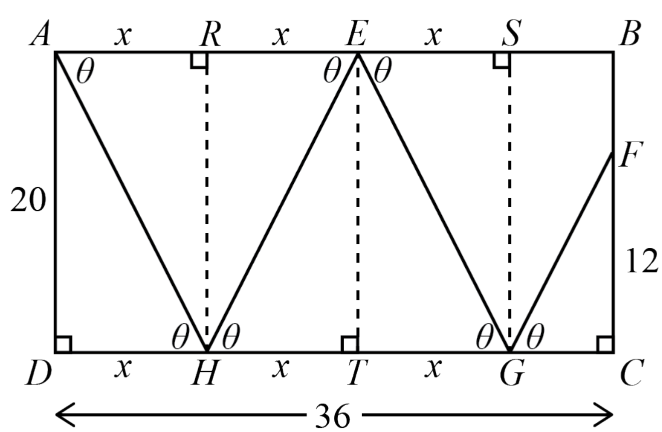


The fence will be constructed so that $\angle AHD = \angle EHG$, $\angle AEH = \angle BEG$, $\angle EGH = \angle FGC$, and $CF = 12$ m. If $AB = 36$ m and $AD = 20$ m, determine the total length of fencing required. That is, determine the value of $AH + EH + EG + FG$.

Solution

Since $ABCD$ is a rectangle, then $AB||CD$. Then $\angle EAH = \angle AHD$, $\angle AEH = \angle EHG$, and $\angle BEG = \angle EGH$. Since $\angle AHD = \angle EHG$, $\angle AEH = \angle BEG$, and $\angle EGH = \angle FGC$, it follows that $\angle EAH = \angle AHD = \angle EHG = \angle AEH = \angle BEG = \angle EGH = \angle FGC = \theta$.

Let R and S be on AB such that RH and SG are perpendicular to AB. Let T be on CD such that ET is perpendicular to CD. Then $\triangle ADH$, $\triangle ARH$, $\triangle ERH$, $\triangle HTE$, $\triangle GTE$, and $\triangle ESG$ all have equal angles and a height of 20 m, so they are all congruent. Let $AR = RE = ES = DH = HT = TG = x$.



Since $\triangle ADH$ and $\triangle FCG$ have equal angles, it follows that they are similar. Then

$$\begin{aligned}\frac{GC}{FC} &= \frac{DH}{AD}\\ \frac{GC}{12} &= \frac{x}{20}\\ GC &= \frac{x}{20} \times 12 = \frac{3x}{5}\end{aligned}$$

Since $DH + HT + TH + GC = 36$, then $x + x + x + \frac{3x}{5} = 36$. Then $\frac{18x}{5} = 36$, so $x = 10$. Then $GC = \frac{3(10)}{5} = 6$. By the Pythagorean Theorem in $\triangle FCG$,

$$\begin{aligned}FG^2 &= FC^2 + GC^2\\ &= 12^2 + 6^2\\ &= 180\end{aligned}$$

Then $FG = \sqrt{180} = 6\sqrt{5}$, since $FG > 0$.

By the Pythagorean Theorem in $\triangle ADH$,

$$\begin{aligned}AH^2 &= AD^2 + DH^2\\ &= 20^2 + 10^2\\ &= 500\end{aligned}$$

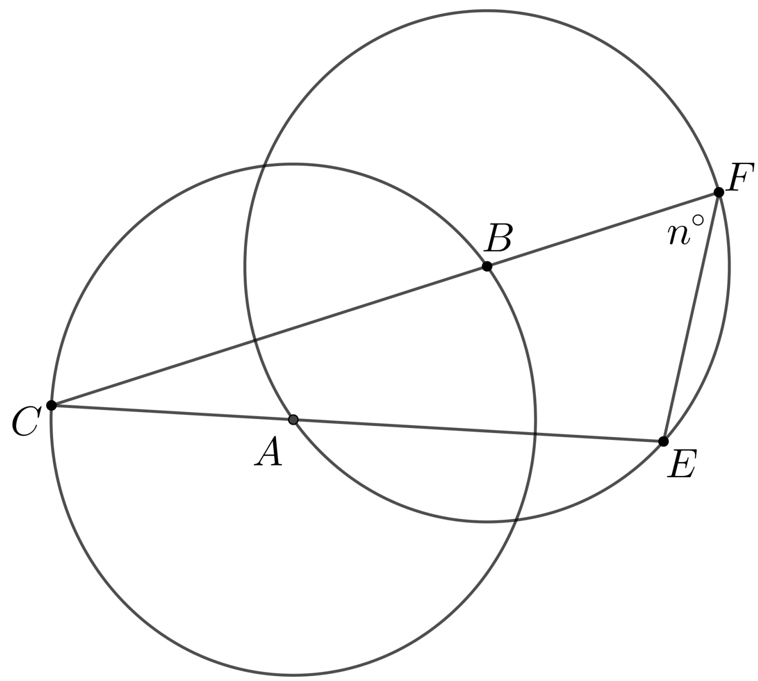
Then $AH = \sqrt{500} = 10\sqrt{5}$, since $AH > 0$.

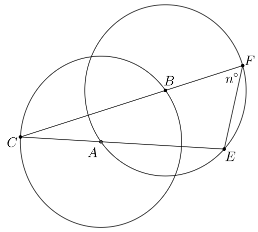
Since $\triangle ADH$, $\triangle HTE$, and $\triangle ETG$ are congruent, it follows that $AH = EH = EG = 10\sqrt{5}$. The total length of fencing required is equal to $AH + EH + EG + FG$, which is $10\sqrt{5} + 10\sqrt{5} + 10\sqrt{5} + 6\sqrt{5} = 36\sqrt{5}$ m.

Problem of the Week
Problem E
Circles and Corners Curiosity

Two circles, with centres A and B, intersect so that A lies on the circle with centre B, and B lies on the circle with centre A. Point C lies on the circle with centre A and points E and F lie on the circle with centre B so that CAE and CBF are straight line segments.

If $\angle CFE = n^\circ$, with $0 < n < 90$, determine the measure of $\angle FCE$ in terms of n.





Problem of the Week
Problem E and Solution
Circles and Corners Curiosity

Problem

Two circles, with centres A and B, intersect so that A lies on the circle with centre B, and B lies on the circle with centre A. Point C lies on the circle with centre A and points E and F lie on the circle with centre B so that CAE and CBF are straight line segments.

If $\angle CFE = n^\circ$, with $0 < n < 90$, determine the measure of $\angle FCE$ in terms of n.

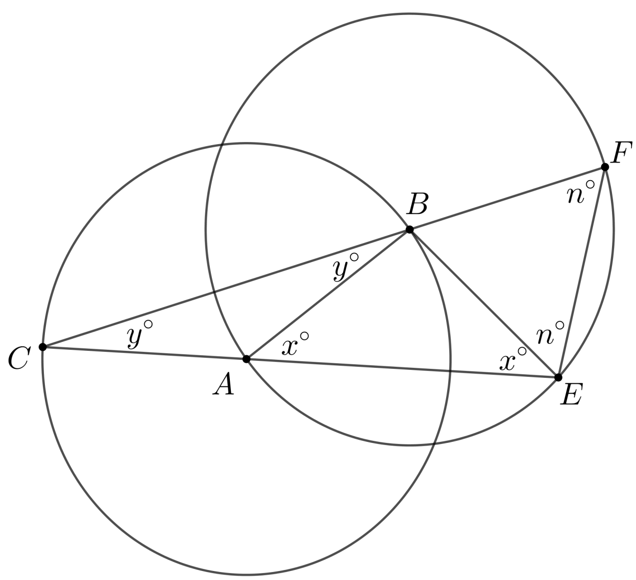
Solution

Draw in AB and BE. Points A, E, and F lie on the circumference of the circle with centre B. Therefore, $BA = BE = BF$. Points B and C lie on the circle with centre A, thus $AB = AC$.

Let $\angle BEA = x^\circ$. Since $BA = BE$, $\triangle BAE$ is isosceles, and so $\angle BAE = \angle BEA = x^\circ$.

Let $\angle ABC = y^\circ$. Since $AB = AC$, $\triangle ABC$ is isosceles, and so $\angle BCA = \angle ABC = y^\circ$.

Also, since $BE = BF$, $\triangle BEF$ is isosceles, and so $\angle BEF = \angle BFE = \angle CFE = n^\circ$.



$\angle BAE$ is an exterior angle to $\triangle ABC$. By the Exterior Angle Theorem for triangles, $\angle BAE = \angle BCA + \angle ABC$. Therefore, $x = 2y$.

In $\triangle CEF$, since the angles in a triangle sum to 180°, we have $\angle FCE + \angle CFE + \angle CEF = 180^\circ$. Since $\angle FCE = \angle BCA = y^\circ$, $\angle CFE = n^\circ$, and $\angle CEF = \angle BEA + \angle BEF = x^\circ + n^\circ$, we have $y^\circ + n^\circ + x^\circ + n^\circ = 180^\circ$. Thus, $y + x + 2n = 180$.

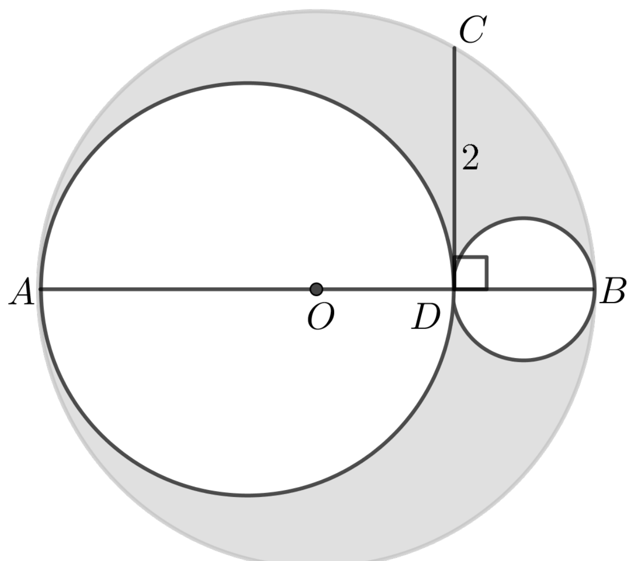
Since $x = 2y$, we have $y + 2y + 2n = 180$, which simplifies to $3y = 180 - 2n$ or $y = 60 - \frac{2}{3}n$.

Therefore, $\angle FCE = y^\circ = \left(60 - \frac{2}{3}n\right)^\circ$.

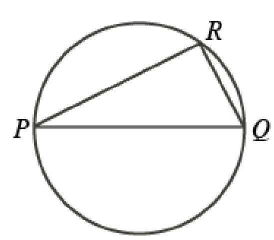
Problem of the Week
Problem E
Embedded Circles

A circle with centre O has diameter AB. A line segment is drawn from a point C on the circumference of the circle to D on OB such that $CD \perp OB$ and $CD = 2$ units. Two circles are drawn on AB. One has diameter AD and the other has diameter DB.

Determine the area of the shaded region. That is, determine the area inside the circle centred at O but outside of the circle with diameter AD and outside of the circle with diameter DB.



NOTE: In solving this problem, it may be helpful to use the fact that the angle inscribed in a circle by the diameter is 90°. For example, in the following diagram, PQ is a diameter and $\angle PRQ$ is inscribed in the circle by diameter PQ. Therefore, $\angle PRQ = 90°$.

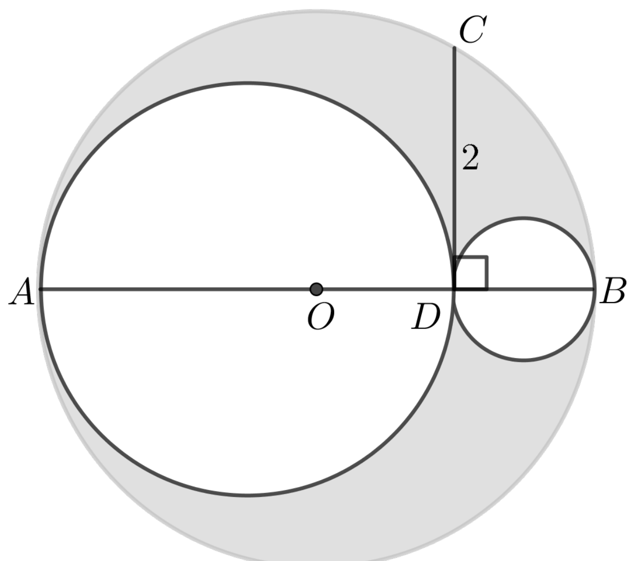


Problem of the Week
Problem E and Solution
Embedded Circles

Problem

A circle with centre O has diameter AB. A line segment is drawn from a point C on the circumference of the circle to D on OB such that $CD \perp OB$ and $CD = 2$ units. Two circles are drawn on AB. One has diameter AD and the other has diameter DB.

Determine the area of the shaded region. That is, determine the area inside the circle centred at O but outside of the circle with diameter AD and outside of the circle with diameter DB.



Solution

Let the radius of the circle with diameter DB be r. Then $DB = 2r$. Let the radius of the circle with diameter AD be R. Then $AD = 2R$.
Also, $AB = AD + DB = 2R + 2r$, and so the radius of the circle with centre O is $R + r$.

It follows that the area of the circle with diameter AD is πR^2, the area of the circle with diameter DB is πr^2, and the area of the circle with centre O is $\pi(R + r)^2$.

To determine the shaded area, we calculate the area of the circle with centre O and subtract the area of the circle with diameter AD and the area of the circle with diameter DB. That is,

$$\begin{aligned}\text{Shaded Area} &= \pi(R+r)^2 - \pi R^2 - \pi r^2 \\ &= \pi(R^2+2Rr+r^2) - \pi R^2 - \pi r^2 \\ &= \pi R^2 + 2\pi Rr + \pi r^2 - \pi R^2 - \pi r^2 \\ &= 2\pi Rr\end{aligned}$$

Join A to C and C to B. Since AB is a diameter and $\angle ACB$ is inscribed in a circle by that diameter, we know that $\angle ACB = 90^\circ$.

Since $CD \perp OB$, then $\angle ODC = \angle ADC = \angle BDC = 90^\circ$. We will use the Pythagorean Theorem in the three triangles $\triangle ADC$, $\triangle BDC$, and $\triangle ACB$, to establish a relationship between R and r.

In $\triangle ADC$, $AC^2 = AD^2 + CD^2 = (2R)^2 + 2^2 = 4R^2 + 4$.

In $\triangle BDC$, $BC^2 = DB^2 + CD^2 = (2r)^2 + 2^2 = 4r^2 + 4$.

In $\triangle ACB$, $AB^2 = AC^2 + BC^2 = (4R^2+4) + (4r^2+4) = 4R^2 + 4r^2 + 8$.

But $AB^2 = (AD + DB)^2 = (2R + 2r)^2 = 4R^2 + 8Rr + 4r^2$.
Therefore, $4R^2 + 8Rr + 4r^2 = 4R^2 + 4r^2 + 8$ and $8Rr = 8$ or $Rr = 1$ follows.

Thus, the shaded area is equal to $2\pi Rr = 2\pi(1) = 2\pi$ units^2.

NOTE: The relationship $Rr = 1$ could also be established using similar triangles as follows:

In $\triangle ADC$, $\angle CAD + \angle ACD = 90^\circ$. Also, since $\angle ACB = 90^\circ$, $\angle ACD + \angle DCB = 90^\circ$.

Thus, $\angle CAD + \angle ACD = \angle ACD + \angle DCB$, which simplifies to $\angle CAD = \angle DCB$.

Now $\angle CAD = \angle DCB$ and $\angle CDA = \angle CDB = 90^\circ$.

Therefore, $\triangle ADC \sim \triangle CDB$ by Angle Angle Angle (AAA) similarity.

From triangle similarity, $\dfrac{AD}{CD} = \dfrac{CD}{DB}$, and so $\dfrac{2R}{2} = \dfrac{2}{2r}$ and $Rr = 1$ follows.

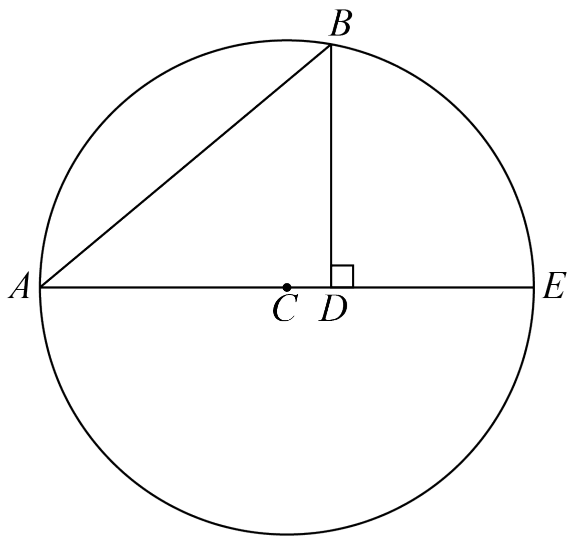
Problem of the Week

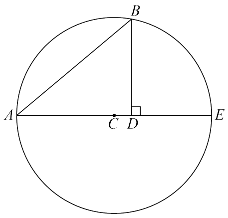
Problem E

One Circle

A circle with centre C has diameter AE. A line segment is drawn from a point B on the circumference of the circle to D on CE such that BD is perpendicular to CE.

If $AB = 24$ and $CD = 2$, determine the radius of the circle.





Problem of the Week

Problem E and Solution

One Circle

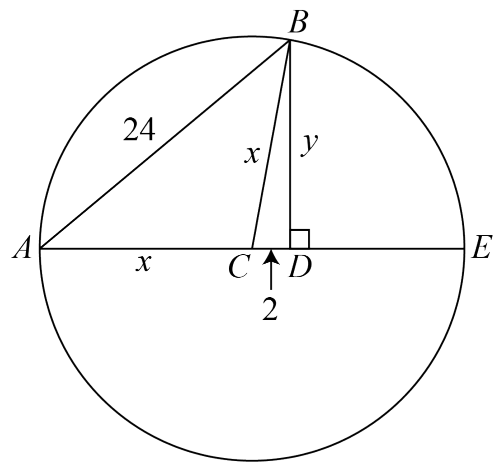
Problem

A circle with centre C has diameter AE. A line segment is drawn from a point B on the circumference of the circle to D on CE such that BD is perpendicular to CE.

If $AB = 24$ and $CD = 2$, determine the radius of the circle.

Solution

Solution 1



Since C is the centre of the circle that passes through A and B, then AC and BC are radii. Let $AC = BC = x$ and $BD = y$.

Since $BD \perp CE$, we have $BD \perp AE$. It follows that $\angle ADB = 90^\circ$. Using the Pythagorean Theorem in $\triangle ADB$,

$$\begin{aligned} (x+2)^2 + y^2 &= 24^2 \\ y^2 &= 576 - (x+2)^2 \qquad (1) \end{aligned}$$

Using the Pythagorean Theorem in $\triangle CDB$,

$$\begin{aligned} 2^2 + y^2 &= x^2 \\ y^2 &= x^2 - 4 \qquad (2) \end{aligned}$$

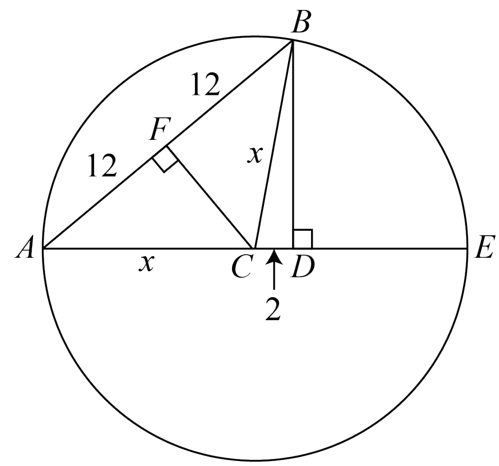
From equations (1) and (2) we can conclude,

$$\begin{aligned} 576 - (x+2)^2 &= x^2 - 4 \\ 576 - x^2 - 4x - 4 &= x^2 - 4 \\ 0 &= 2x^2 + 4x - 576 \\ 0 &= x^2 + 2x - 288 \\ 0 &= (x+18)(x-16) \end{aligned}$$

Thus, $x = 16$, since $x > 0$. Therefore, the radius of the circle is 16.

Solution 2

Since C is the centre of the circle that passes through A and B, then AC and BC are radii. Let $AC = BC = x$. Let F be the point on AB such that CF is perpendicular to AB. Since $\triangle ABC$ is isosceles, F is also the midpoint of AB.



Since $\angle BAC = \angle BAD$ and $\angle AFC = \angle ADB = 90°$, it follows that $\triangle AFC \sim \triangle ADB$. Therefore,

$$\begin{aligned}\frac{AC}{AF} &= \frac{AB}{AD}\\ \frac{x}{12} &= \frac{24}{x+2}\\ x(x+2) &= 288\\ x^2 + 2x - 288 &= 0\\ (x+18)(x-16) &= 0\end{aligned}$$

Thus, $x = 16$, since $x > 0$. Therefore, the radius of the circle is 16.

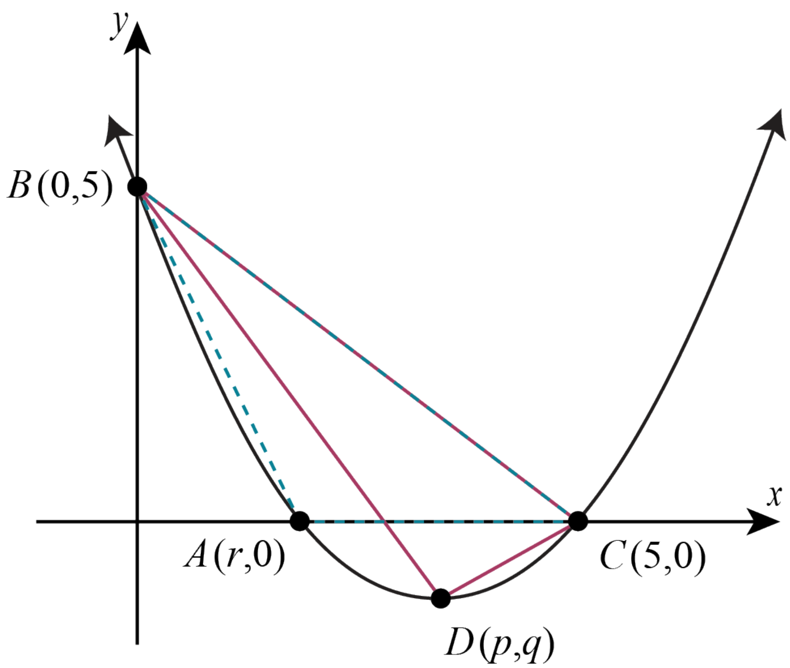
Problem of the Week

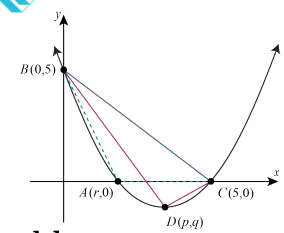
Problem E

No Parabolema to Find the Area!

A parabola intersects the y-axis at $B(0,5)$, and intersects the x-axis at $C(5,0)$ and at $A(r,0)$, where $0 < r < 5$. The area of $\triangle ABC$ is 5 units^2.

If $D(p,q)$ is the vertex of the parabola, then determine the area of $\triangle DBC$.





Problem of the Week
Problem E and Solution
No Parabolema to Find the Area!

Problem

A parabola intersects the y-axis at $B(0,5)$, and intersects the x-axis at $C(5,0)$ and at $A(r,0)$, where $0 < r < 5$. The area of $\triangle ABC$ is 5 units^2.

If $D(p,q)$ is the vertex of the parabola, then determine the area of $\triangle DBC$.

Solution

The height of $\triangle ABC$ is the distance from the x-axis to $B(0,5)$, which is 5 units. The base is $AC = 5 - r$. Since the area of $\triangle ABC$ is 5, using the formula for the area of a triangle, we have $\frac{(5-r)(5)}{2} = 5$. Then $5 - r = 2$ and $r = 3$ follows. Thus, the coordinates of A are $(3,0)$.

The axis of symmetry of the parabola is a vertical line through the midpoint of AC, which is $(4,0)$. It follows that the x-coordinate of the vertex is $p = 4$. Therefore, the vertex is $D(4,q)$.

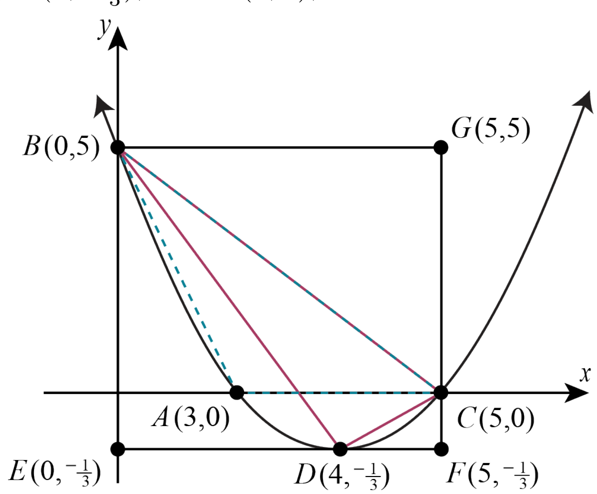
Since the two x-intercepts of the parabola are 3 and 5, the equation of the parabola in factored form can be written as $y = a(x-3)(x-5)$. Since the parabola passes through $B(0,5)$, we can solve for a by substituting $x = 0$ and $y = 5$ into $y = a(x-3)(x-5)$. This leads to $a = \frac{1}{3}$ and thus the parabola has equation $y = \frac{1}{3}(x-3)(x-5)$.

To determine q, the y-coordinate of D, we substitute $x = 4$, $y = q$ into $y = \frac{1}{3}(x-3)(x-5)$. Then $q = \frac{1}{3}(4-3)(4-5) = -\frac{1}{3}$. Therefore, D has coordinates $(4, -\frac{1}{3})$.

From here, we proceed with two different solutions to determine the area of $\triangle DBC$.

Solution 1

Consider points $E(0,-\frac{1}{3})$, $F(5,-\frac{1}{3})$, and $G(5,5)$, and draw in $BGFE$.



Since B and G have the same y-coordinate, BG is a horizontal line. Since G and F both have x-coordinate 5, GF is a vertical line which passes through C. Since E and F both have y-coordinate $-\frac{1}{3}$, EF is a horizontal line which passes through D. Since B and E have the same x-coordinate, BE is a vertical line. Thus, $BGFE$ is a rectangle that encloses $\triangle DBC$, and we have

$$\text{area } \triangle DBC = \text{area } BGFE - \text{area } \triangle BGC - \text{area } \triangle DFC - \text{area } \triangle BED$$

In rectangle $BGFE$, $BG = 5 - 0 = 5$ and $BE = 5 - (-\frac{1}{3}) = \frac{16}{3}$. The area of rectangle $BGFE = BG \times BE = 5 \times \frac{16}{3} = \frac{80}{3}$ units^2.

Since $BGFE$ is a rectangle, $\triangle BGC$ is right-angled at G. Since $BG = 5$ and $GC = 5 - 0 = 5$, the area of $\triangle BGC = \frac{BG \times GC}{2} = \frac{5 \times 5}{2} = \frac{25}{2}$ units^2.

Since $BGFE$ is a rectangle, $\triangle DFC$ is right-angled at F. Since $CF = 0 - (-\frac{1}{3}) = \frac{1}{3}$ and $DF = 5 - 4 = 1$, the area of $\triangle DFC = \frac{CF \times DF}{2} = \frac{\frac{1}{3} \times 1}{2} = \frac{1}{6}$ units^2.

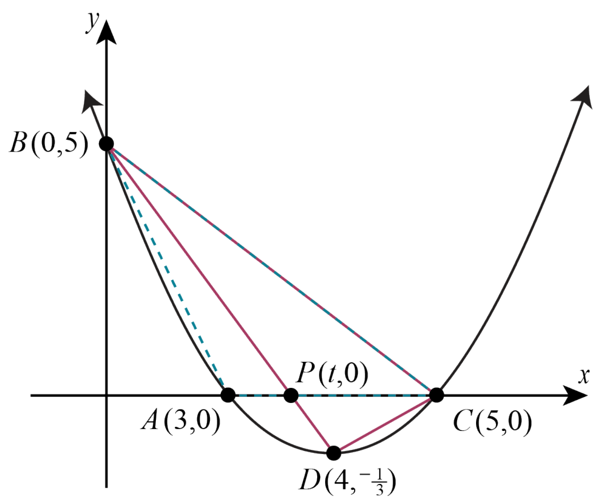
Since $BGFE$ is a rectangle, $\triangle BED$ is right-angled at E. Since $BE = \frac{16}{3}$ and $ED = 4 - 0 = 4$, the area of $\triangle BED = \frac{BE \times ED}{2} = \frac{\frac{16}{3} \times 4}{2} = \frac{32}{3}$ units^2.

Thus,

$$\begin{aligned} \text{area } \triangle DBC &= \text{area } BGFE - \text{area } \triangle BGC - \text{area } \triangle DFC - \text{area } \triangle BED \\ &= \tfrac{80}{3} - \tfrac{25}{2} - \tfrac{1}{6} - \tfrac{32}{3} \\ &= \tfrac{10}{3} \text{ units}^2 \end{aligned}$$

Solution 2

Let $P(t, 0)$ be the point where the line through B and D crosses the x-axis. We will determine the equation of the line that passes through B, P, and D.



Since the line passes through $B(0, 5)$ and $D(4, -\frac{1}{3})$, the slope of the line is $\frac{5+\frac{1}{3}}{0-4} = \frac{\frac{16}{3}}{-4} = -\frac{4}{3}$.

The y-intercept of the line is 5. Therefore, the equation of the line through B, P, and D is $y = -\frac{4}{3}x + 5$.

To determine t, the x-coordinate of P we substitute $x = t$ and $y = 0$ into $y = -\frac{4}{3}x + 5$, the equation of the line. Thus, $0 = -\frac{4}{3}t + 5$, and $4t = 15$ or $t = \frac{15}{4}$ follows.

In $\triangle BPC$, the height is the perpendicular distance from the x-axis to point B, which is 5. The base is $PC = 5 - \frac{15}{4} = \frac{5}{4}$. Thus, the area of $\triangle BPC = \frac{\frac{5}{4} \times 5}{2} = \frac{25}{8}$ units^2.

In $\triangle DPC$, the height is the perpendicular distance from the x-axis to point D, which is $\frac{1}{3}$. The base is $PC = 5 - \frac{15}{4} = \frac{5}{4}$. Thus, the area of $\triangle DPC = \frac{\frac{5}{4} \times \frac{1}{3}}{2} = \frac{5}{24}$ units^2.

Therefore, the area of $\triangle DBC = \text{area } \triangle BPC + \text{area } \triangle DPC = \frac{25}{8} + \frac{5}{24} = \frac{10}{3}$ units^2.

Number Sense (N)

Take me to the cover

Problem of the Week

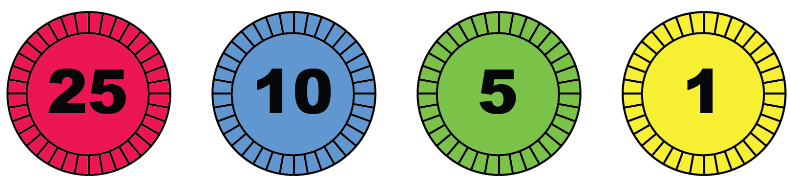
Problem E

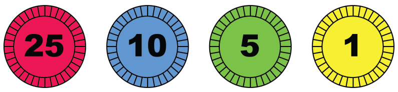
Exactly Four

A machine dispenses four kinds of tokens. Red tokens are worth 25 points, blue tokens are worth 10 points, green tokens are worth 5 points, and yellow tokens are worth 1 point. The machine never runs out of tokens to dispense.

To get tokens from the machine, people enter a total number of points, and the machine will dispense the fewest number of tokens that total that number of points. For example, if someone enters 30 points, the machine will dispense 1 red token and 1 green token, because their total value is $25 + 5 = 30$ points, and this total cannot be obtained using fewer tokens.

For how many different total numbers of points will the machine dispense exactly 4 tokens?





Problem of the Week
Problem E and Solution
Exactly Four

Problem

A machine dispenses four kinds of tokens. Red tokens are worth 25 points, blue tokens are worth 10 points, green tokens are worth 5 points, and yellow tokens are worth 1 point. The machine never runs out of tokens to dispense.

To get tokens from the machine, people enter a total number of points, and the machine will dispense the fewest number of tokens that total that number of points. For example, if someone enters 30 points, the machine will dispense 1 red token and 1 green token, because their total value is $25 + 5 = 30$ points, and this total cannot be obtained using fewer tokens.

For how many different total numbers of points will the machine dispense exactly 4 tokens?

Solution

We start by noting how many of the 4 tokens can be a particular colour. There can be up to 4 red tokens because they have the largest value, so the machine cannot obtain the same total number of points using fewer tokens. There can also be up to 4 yellow tokens, because 4 yellow tokens have a total value of 4 points, which is less than the value of 1 green token. The number of blue tokens must be less than 3, because 3 blue tokens have a total value of 30 points, which can be obtained using fewer tokens (namely 1 red token and 1 green token). Similarly the number of green tokens must be less than 2, because 2 green tokens have a total value of 10 points, which is the value of 1 blue token. Finally, we cannot have 2 blue tokens and 1 green token, because they have a total value of 25 points, which is the value of 1 red token.

Now we count the number of combinations of 4 tokens that can be dispensed from the machine.

- **Case 1:** There are 4 red tokens.
  In this case, the only total number of points possible is 100.
- **Case 2:** There are 3 red tokens.
  If there are 3 red tokens, then the other token could be blue, green, or yellow. The total numbers of points would then be 85, 80, or 76, respectively. Thus, there are 3 different total numbers of points.
- **Case 3:** There are 2 red tokens.
  If there are 2 red tokens, then the other 2 tokens are either both the same colour, or are two different colours.
  - If the other 2 tokens are both the same colour, then they can either be blue or yellow. The total numbers of points would then be 70 or 52, respectively. Note that they cannot be green because the number of green tokens must be less than 2.
  - If the other 2 tokens are two different colours, then they can be blue and green, blue and yellow, or green and yellow. The total numbers of points would then be 65, 61, or 56, respectively.

  Thus, there are 5 different total numbers of points if there are 2 red tokens.

- **Case 4:** There is 1 red token.
  If there is 1 red token, then the other 3 tokens are either all the same colour, 2 are the same colour and 1 is a different colour, or they are all different colours.
  - If the other 3 tokens are all the same colour, then they must all be yellow. Therefore, the total number of points would be 28. Note that they cannot all be blue or all be green because the number of blue tokens must be less than 3 and the number of green tokens must be less than 2.
  - If 2 of the other 3 tokens are the same colour and 1 is a different colour, then there could be 2 blue and 1 yellow, 1 blue and 2 yellow, or 1 green and 2 yellow. The total numbers of points would then be 46, 37, or 32, respectively. Note that we already determined that we cannot have 2 blue tokens and 1 green token.
  - If the other 3 tokens are all different colours, then the total number of points would be 41.

  Thus, there are 5 different total numbers of points if there is 1 red token.

- **Case 5:** There are no red tokens.
  If there are no red tokens, then there are either 4 tokens of one colour, 3 tokens of one colour and 1 token of another colour, 2 tokens of one colour and 2 tokens of another colour, or 2 tokens of one colour and 1 token of each of the other two colours.
  - If there are 4 tokens of one colour, then they must be all yellow. Therefore, the total number of points would be 4.
  - If there are 3 tokens of one colour and 1 token of another colour, then there could be 1 blue token and 3 yellow tokens or 1 green token and 3 yellow tokens. The total numbers of points would then be 13 or 8, respectively.
  - If there are 2 tokens of one colour and 2 tokens of another colour, then there must be 2 blue tokens and 2 yellow tokens. Therefore, the total number of points would be 22.
  - If there are 2 tokens of one colour and 1 token of each of the other two colours, then there must be 1 blue token, 1 green token, and 2 yellow tokens. Therefore, the total number of points would be 17.

  Thus, there are 5 different total numbers of points if there are no red tokens.

In total, there are $1 + 3 + 5 + 5 + 5 = 19$ different total numbers of points for which the machine will dispense exactly 4 tokens.

The 19 possible totals obtained using exactly 4 tokens are as follows.

$$4,\ 8,\ 13,\ 17,\ 22,\ 28,\ 32,\ 37,\ 41,\ 46,\ 52,\ 56,\ 61,\ 65,\ 70,\ 76,\ 80,\ 85,\ 100$$

Problem of the Week

Problem E

Fraction Distraction

Determine the number of solutions to the equation

$$\frac{A}{B} - \frac{B}{A} = \frac{A+B}{AB}$$

where A and B are both integers, $-9 \le A \le 9$, and $-9 \le B \le 9$.

$\frac{A}{B} - \frac{B}{A} = \frac{A+B}{AB}$

Problem of the Week
Problem E and Solution
Fraction Distraction

Problem

Determine the number of solutions to the equation

$$\frac{A}{B} - \frac{B}{A} = \frac{A+B}{AB}$$

where A and B are both integers, $-9 \leq A \leq 9$, and $-9 \leq B \leq 9$.

Solution

First notice that neither A nor B can equal zero. Starting with the equation, we simplify as follows.

$$\begin{aligned} \frac{A}{B} - \frac{B}{A} &= \frac{A+B}{AB} \\ \frac{A^2}{AB} - \frac{B^2}{AB} &= \frac{A+B}{AB} \\ \frac{A^2 - B^2}{AB} &= \frac{A+B}{AB} \\ \frac{(A-B)(A+B)}{AB} &= \frac{A+B}{AB} \\ (A-B)(A+B) &= A+B \end{aligned}$$

Since the two sides are equal, $A - B = 1$ or $A + B = 0$. We will consider these two cases.

Case 1: $A - B = 1$

In this case, we know that A and B differ by 1 and $A > B$. The largest value A can be is 9. When $A = 9$, $B = 8$. The smallest value B can be is -9. When $B = -9$, $A = -8$, a value which is 1 more than the value of B.

So A can take on all of the integer values from -8 to 9, except $A = 0$. But when $A = 1$, $B = 0$, so we have to remove this value of A as well. There are 18 values for A from -8 to 9. After removing $A = 0$ and $A = 1$, there are 16 values for A and therefore 16 corresponding values for B. Thus, the equation has 16 solutions when $A - B = 1$.

Case 2: $A + B = 0$

In this case, $A + B = 0$ or $A = -B$. The largest value A can be is 9. When $A = 9$, $B = -9$. The smallest value A can be is -9. When $A = -9$, $B = 9$. So A can take on all of the integer values from -9 to 9, except $A = 0$. Thus, the equation has $19 - 1 = 18$ solutions when $A + B = 0$.

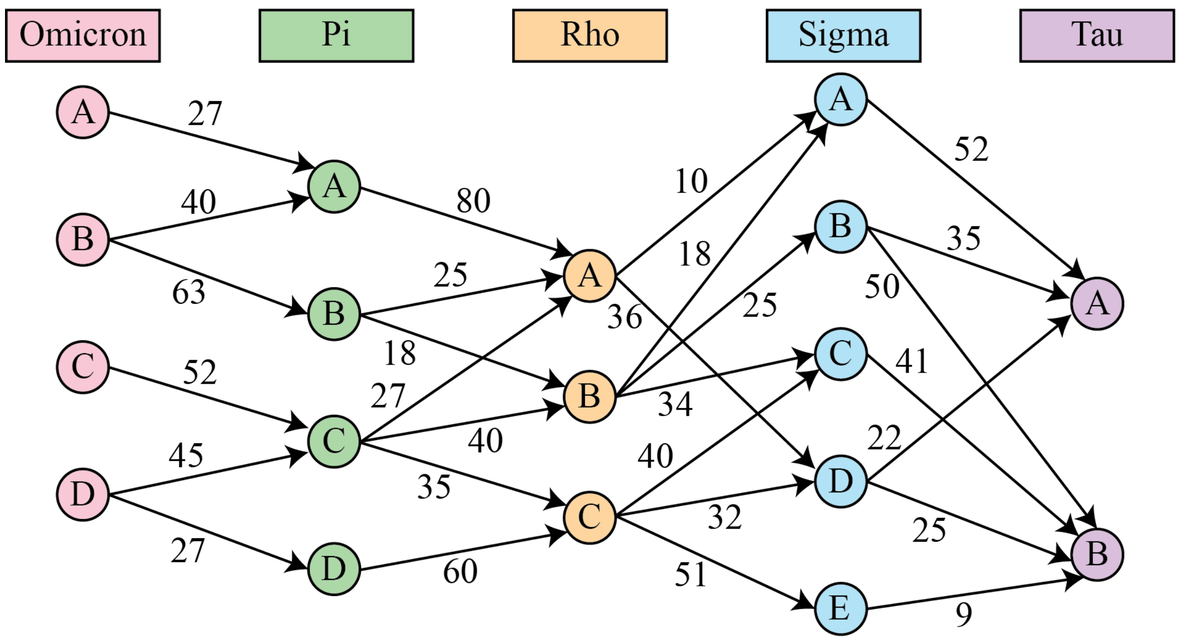
Therefore, there are $16 + 18 = 34$ solutions to the equation which satisfy the conditions that A and B are both integers, $-9 \leq A \leq 9$, and $-9 \leq B \leq 9$.

Problem of the Week

Problem E

The Fantastic Race

As part of The Fantastic Race, teams need to travel on buses from city to city in the order Omicron to Pi to Rho to Sigma to Tau. Each city has several different bus stations to choose from. Nate has created the following map showing all the different bus routes between the five cities, as well as the travel time, in minutes, for each. The different bus stations within each city are labeled A, B, C, etc.



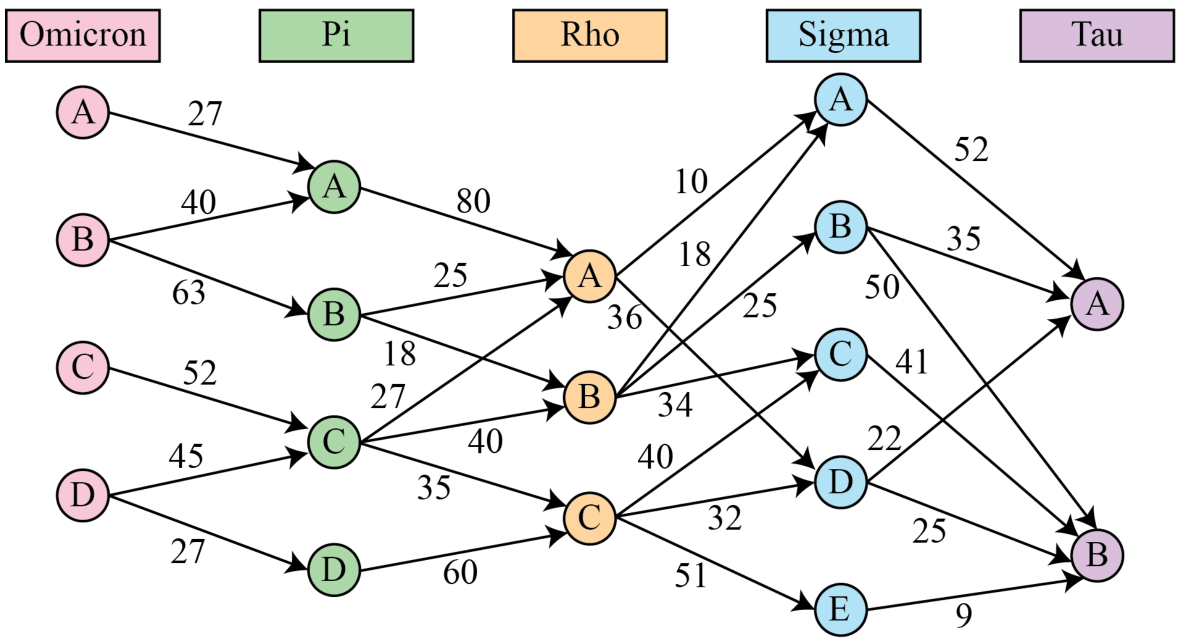
Which route from Omicron to Tau gives the shortest total travel time?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Problem of the Week
Problem E and Solution
The Fantastic Race

Problem

As part of The Fantastic Race, teams need to travel on buses from city to city in the order Omicron to Pi to Rho to Sigma to Tau. Each city has several different bus stations to choose from. Nate has created the following map showing all the different bus routes between the five cities, as well as the travel time, in minutes, for each. The different bus stations within each city are labeled A, B, C, etc.



Which route from Omicron to Tau gives the shortest total travel time?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Solution

The shortest total travel time is 130 min, and can be achieved with the following route: Omicron (D) → Pi (C) → Rho (A) → Sigma (D) → Tau (A).

In order to find this route, we use a method called *dynamic programming*. In dynamic programming, we systematically build the solution from small pieces to bigger and bigger pieces. This saves us a significant amount of time because we don't have to calculate the total travel time for every possible route.

We start by looking at the routes from Omicron to Pi, and determine the route with the shortest travel time to each of the bus stations in Pi. We will call these the "best" routes. The best route to reach Pi (A) takes 27 min and comes from

Omicron (A). The only route to reach Pi (B) takes 63 min and comes from Omicron (B). The best route to reach Pi (C) takes 45 min and comes from Omicron (D). The only route to reach Pi (D) takes 27 min and comes from Omicron (D). We then ignore all other routes from Omicron to Pi.

Next we look at the routes from Omicron to Rho, using the information we just recorded. The best route to reach Rho (A) takes $45 + 27 = 72$ min and comes from Pi (C). The best route to reach Rho (B) takes $63 + 18 = 81$ min and comes from Pi (B). The best route to reach Rho (C) takes $45 + 35 = 80$ min and comes from Pi (C).

Next we look at the routes from Omicron to Sigma, using the information we just recorded. The best route to reach Sigma (A) takes $72 + 10 = 82$ min and comes from Rho (A). The best route to reach Sigma (B) takes $81 + 25 = 106$ min and comes from Rho (B). The best route to reach Sigma (C) takes $81 + 34 = 115$ min and comes from Rho (B). The best route to reach Sigma (D) takes $72 + 36 = 108$ min and comes from Rho (A). The best route to reach Sigma (E) takes $80 + 51 = 131$ min and comes from Rho (C).

Finally, we look at the routes from Omicron to Tau, using the information we just recorded. The best route to reach Tau (A) takes $108 + 22 = 130$ min and comes from Sigma (D). The best route to reach Tau (B) takes $108 + 25 = 133$ min and comes from Sigma (D).

Thus, the shortest total travel time is 130 min, and is achieved with the route Omicron (D) $\rightarrow$ Pi (C) $\rightarrow$ Rho (A) $\rightarrow$ Sigma (D) $\rightarrow$ Tau (A).

Problem of the Week

Problem E

Summing up a Sequence 2

The first term in a sequence is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

Shweta writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. How many different possible values of n are there?

24, 12, 6, ...

24, 12, 6, ...

Problem of the Week
Problem E and Solution
Summing up a Sequence 2

Problem

The first term in a sequence is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

Shweta writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. How many different possible values of n are there?

Solution

We will begin by finding more terms in the sequence. The first 14 terms of the sequence are 24, 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1.

If we continue the sequence, we will see that the terms 4, 2, and 1 will continue to repeat. Now we want to find the smallest and largest possible values of n so that the sum of the terms in the sequence from term 1 to term n is a four-digit number. We will start by finding the smallest possible value of n.

The sum of the first 8 terms is $24 + 12 + 6 + 3 + 10 + 5 + 16 + 8 = 84$. The sum of the repeating numbers is $4 + 2 + 1 = 7$. We want to determine the number of groups of repeating numbers. Let this be g. Suppose $84 + 7g = 1000$. Solving this gives $7g = 916$, so $g \approx 130.857$.

If $g = 130$, then the sum of the terms in the sequence is $84 + 7 \times 130 = 994$. This sequence contains the first 8 terms, plus 130 groups of the three repeating numbers. Therefore there are a total of $8 + 3 \times 130 = 398$ terms.

The 399^{th} term in the sequence will be 4, so the sum of the first 399 terms will be $994 + 4 = 998$.

The 400^{th} term in the sequence will be 2, so the sum of the first 400 terms will be $998 + 2 = 1000$. This is the smallest possible four-digit number, so the smallest possible value of n is 400.

Now we will find the largest possible value of n. Using a similar approach, let g be the number of groups of repeating numbers. Suppose $84 + 7g = 9999$. Solving this gives $g \approx 1416.429$.

If $g = 1416$, then the sum of the terms in the sequence is $84 + 7 \times 1416 = 9996$. This sequence contains the first 8 terms, plus 1416 groups of the three repeating numbers. Therefore there are a total of $8 + 3 \times 1416 = 4256$ terms.

The 4257^{th} term in the sequence will be 4, so the sum of the first 4257 terms will be $9996 + 4 = 10\,000$. Since this is not a four-digit number, the largest possible value of n is 4256.

So n can be any positive integer between 400 and 4256, inclusive. This is a total of $4256 - 400 + 1 = 3857$ possible values.

Extension:

In 1937, the mathematician Lothat Collatz wondered if any sequence whose terms after the first are determined in this way would always eventually reach the number 1, regardless of which number you started with. This problem is actually still unsolved today and is called the Collatz Conjecture.

Problem of the Week

Problem E

A Geometric Problem

The first term in a geometric sequence is a, the second term is b, and the third term is c. The three terms have a sum of 158 and a product of 74 088.

Determine all possible ordered triples (a, b, c).

$$t_n = ar^{n-1}$$

NOTE: The general term of a geometric sequence can be written as $t_n = ar^{n-1}$, where a is first term of the sequence, r is the common ratio between terms, and t_n is the n^{th} term.

Problem of the Week

$t_n = ar^{n-1}$

Problem E and Solution

A Geometric Problem

Problem

The first term in a geometric sequence is a, the second term is b, and the third term is c. The three terms have a sum of 158 and a product of 74 088.

Determine all possible ordered triples (a, b, c).

Solution

Let r be the common ratio of the geometric sequence. Since a is the first term of the sequence, then $b = ar$ and $c = ar^2$.

We are given that $abc = 74\,088$. Thus, $a(ar)(ar^2) = a^3r^3 = (ar)^3 = 74\,088$. Therefore, $ar = 42$. Since $b = ar$, we have $b = 42$.

Now, $a + b + c = 158$ becomes $a + 42 + c = 158$, or $a + c = 116$.

Since $b = ar$, then $42 = ar$, or $r = \dfrac{42}{a}$ (since the product of a, b, and c is not zero, we know $a \neq 0$).

Therefore, $c = ar^2 = a\left(\dfrac{42}{a}\right)^2 = a\left(\dfrac{1764}{a^2}\right) = \dfrac{1764}{a}$.

Substituting $c = \dfrac{1764}{a}$ into $a + c = 116$, we have

$$\begin{aligned} a + \frac{1764}{a} &= 116 \\ a^2 + 1764 &= 116a \\ a^2 - 116a + 1764 &= 0 \\ (a-18)(a-98) &= 0 \end{aligned}$$

Therefore, $a = 18$ or $a = 98$.

When $a = 18$, then $r = \frac{42}{18} = \frac{7}{3}$, and one ordered triple is (18, 42, 98). Indeed, we can check that $18 + 42 + 98 = 158$ and $(18)(42)(98) = 74\,088$.

When $a = 98$, then $r = \frac{42}{98} = \frac{3}{7}$, and one ordered triple is (98, 42, 18). Indeed, we can check that $98 + 42 + 18 = 158$ and $(98)(42)(18) = 74\,088$.

In conclusion there are two ordered triples that satisfy the conditions of the problem. They are (18, 42, 98) and (98, 42, 18).

Problem of the Week

Problem E

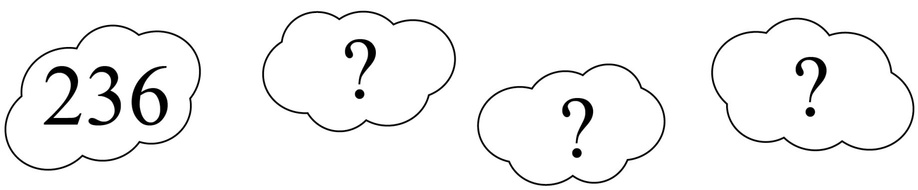
Four Numbers

Norbert has four favourite numbers. Each of these is a three-digit number ABC with the following two properties:

1. The digits A, B, and C are all different.
2. The product $A \times B \times C$ is equal to the two-digit number BC.

For example, one of Norbert's favourite numbers is 236, since $2 \times 3 \times 6 = 36$.

Find Norbert's other three favourite numbers.

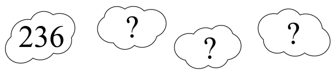


Note: It may be helpful to recall that any two-digit number of the form BC can be represented by the sum $10B + C$. For example, $32 = 10(3) + 2$.

Problem of the Week

Problem E and Solution

Four Numbers



Problem

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1. The digits A, B, and C are all different.
2. The product $A \times B \times C$ is equal to the two-digit number BC.

For example, one of Norbert's favourite numbers is 236, since $2 \times 3 \times 6 = 36$.

Find Norbert's other three favourite numbers.

Note: It may be helpful to recall that any two-digit number of the form BC can be represented by the sum $10B + C$. For example, $32 = 10(3) + 2$.

Solution

We know that there are only four answers, so we could attempt a trial and error approach to find the remaining three numbers, or even write a program that could find them using a brute force approach. However here we will present a more systematic approach.

First, since A, B, and C are digits, they must be positive integers between 0 and 9, inclusive. Also, since the product $A \times B \times C$ is a two-digit number, none of A, B, or C can equal zero.

We want to find all three-digit numbers ABC such that $A \times B \times C = 10B + C$. Since $B \neq 0$, we can divide by B and the problem becomes equivalent to finding all integers A, B, C with $1 \leq A, B, C \leq 9$ and A, B, and C distinct such that

$$A \times C = 10 + \frac{C}{B}$$

Since A and C are integers, then so is $A \times C$, so it follows that $\frac{C}{B}$ must be an integer as well. Therefore, for each possible value of C, we must have that B divides exactly into C.

We will break the problem into cases based on the value of C, and then sub-cases based on the possible values of B.

- **Case 1:** $C = 1$
  There are no values of B where $\frac{C}{B}$ is an integer and $B \neq C$.

- **Case 2:** $C = 2$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$. If $B = 1$ and $C = 2$, $A \times B \times C = 10B + C$ becomes $2A = 12$ and so $A = 6$. Therefore, one of the three-digit numbers is 612.

- **Case 3:** $C = 3$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$. If $B = 1$ and $C = 3$, $A \times B \times C = 10B + C$ becomes $3A = 13$ and so $A = \frac{13}{3}$. Since A is not an integer, there is no solution in this case.

- **Case 4:** $C = 4$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$ or $B = 2$.
  - **Case 4a:** $B = 1$
    In this case, $A \times B \times C = 10B + C$ becomes $4A = 14$ and so $A = \frac{7}{2}$. Since A is not an integer, there is no solution in this case.
  - **Case 4b:** $B = 2$
    In this case, $A \times B \times C = 10B + C$ becomes $8A = 24$ and so $A = 3$. Therefore, one of the three-digit numbers is 324.
- **Case 5:** $C = 5$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$. If $B = 1$ and $C = 5$, $A \times B \times C = 10B + C$ becomes $5A = 15$ and so $A = 3$. Therefore, one of the three-digit numbers is 315.
- **Case 6:** $C = 6$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$, $B = 2$, or $B = 3$.
  - **Case 6a:** $B = 1$
    In this case, $A \times B \times C = 10B + C$ becomes $6A = 16$ and so $A = \frac{8}{3}$. Since A is not an integer, there is no solution in this case.
  - **Case 6b:** $B = 2$
    In this case, $A \times B \times C = 10B + C$ becomes $12A = 26$ and so $A = \frac{13}{6}$. Since A is not an integer, there is no solution in this case.
  - **Case 6c:** $B = 3$
    In this case, $A \times B \times C = 10B + C$ becomes $18A = 36$ and so $A = 2$. Therefore, one of the three-digit numbers is 236. This is the number given in the example.

We can actually stop here since we have found 4 different three-digit numbers that satisfy the conditions outlined in the problem. If we had not been given the number of possible solutions, we would need to continue by checking cases when $C = 7$, $C = 8$, and $C = 9$. It turns out that there are no solutions in these cases.

Therefore, Norbert's four favourite numbers are 612, 324, 315, and 236.

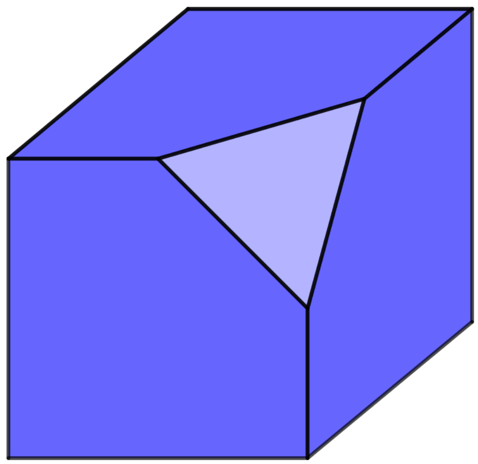
Problem of the Week

Problem E

Cutting Corners

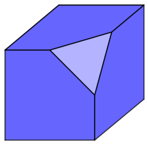
Edwin has a cube with edge length 8 cm. He cuts off a corner by doing the following steps:

1. He finds the midpoint of each edge.
2. He then makes a cut through three of these points on adjacent edges.



He then removes the other seven corners by making similar cuts.

Edwin thinks that the new shape will have a smaller total surface area than the original cube. Show that Edwin is right by finding how much less the new surface area is.



Problem of the Week
Problem E and Solution
Cutting Corners

Problem

Edwin has a cube with edge length 8 cm. He cuts off a corner by doing the following steps:

1. He finds the midpoint of each edge.
2. He then makes a cut through three of these points on adjacent edges.

He then removes the other seven corners by making similar cuts.

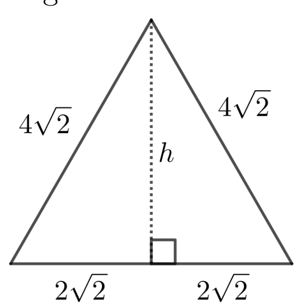
Edwin thinks that the new shape will have a smaller total surface area than the original cube. Show that Edwin is right by finding how much less the new surface area is.

Solution

We will consider one corner, determine the surface area decrease there, and then multiply the result by 8 to account for the eight corners. At each corner, since the cut is made through the midpoints of three adjacent edges of the cube, the surface areas of three identical isosceles right-angled triangles, each with 4 cm base and 4 cm height, are removed and replaced by the surface area of a single equilateral triangle.

Since the isosceles right-angled triangles each have 4 cm base and 4 cm height, each has area equal to $\frac{1}{2}(4)(4) = 8 \text{ cm}^2$.

Each side length of the equilateral triangle is formed by the hypotenuse of one of the isosceles right-angled triangles. Using the Pythagorean Theorem, we calculate the length of the hypotenuse of each right-angled triangle to be $\sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$ cm. Thus, the remaining equilateral triangle has sides of length $4\sqrt{2}$ cm. Let h be the height of the equilateral triangle. Since the triangle is equilateral, the height bisects the base.



By the Pythagorean Theorem, $h^2 = (4\sqrt{2})^2 - (2\sqrt{2})^2 = 32 - 8 = 24$. Since $h > 0$, we have $h = \sqrt{24} = 2\sqrt{6}$ cm.

Therefore, the area of the remaining equilateral triangle is $\frac{1}{2}(4\sqrt{2})(2\sqrt{6}) = 4\sqrt{12} = 8\sqrt{3} \text{ cm}^2$.

At each corner, the surface area is increased by the area of the equilateral triangle and decreased by the areas of the three right-angled triangles. Therefore, removing a corner changes the surface area by $8\sqrt{3} - 3(8) = (8\sqrt{3} - 24) \text{ cm}^2$. Since $8\sqrt{3} < 24$, this result is negative and the surface area is decreased in each corner. Therefore, removing a corner decreases the surface area by $(24 - 8\sqrt{3}) \text{ cm}^2$.

Since there are eight corners, the total decrease in surface area is

$$8 \times (24 - 8\sqrt{3}) = 192 - 64\sqrt{3} \doteq 81.1 \text{ cm}^2$$

Problem of the Week

Problem E

Everything but a Seven

The integer 107 contains the digit 7. The integer 358 does not contain the digit 7. Determine the sum of all integers from 1 to 2024 that do not contain the digit 7.

358 107

Problem of the Week

Problem E and Solution

Everything but a Seven

Problem

The integer 107 contains the digit 7. The integer 358 does not contain the digit 7.

Determine the sum of all integers from 1 to 2024 that do not contain the digit 7.

Solution

Solution 1

In this solution, we will use the fact that the sum of the integers from 1 to n, where n is some positive integer, is $\frac{n(n+1)}{2}$. Consider the integers from 1 to 100. The sum of these integers is $\frac{(100)(101)}{2} = 5050$.

The integers from 1 to 100 which do contain the digit 7 are 7, 17, 27, 37, 47, 57, 67, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 87, 97. The sum of these 19 integers is 1188. Therefore, the sum of the $100 - 19 = 81$ integers from 1 to 100 which do not contain the digit 7 is $5050 - 1188 = 3862$.

There are 81 integers from 101 to 200 which do not contain the digit 7 as well. Each of these is 100 more than a corresponding integer between 1 and 100 which does not contain the digit 7. Thus, the sum of these 81 integers is $3862 + 81(100)$.

We can use this approach to determine the sum of the integers which do not contain the digit 7 in various ranges of 100. This is summarized in the table below.

Range of integers	Number of integers in range that do not contain a 7	Sum of integers in range that do not contain a 7
1 to 100	81	3862
101 to 200	81	$3862 + 81(100)$
201 to 300	81	$3862 + 81(200)$
301 to 400	81	$3862 + 81(300)$
401 to 500	81	$3862 + 81(400)$
501 to 600	81	$3862 + 81(500)$
601 to 700	81	$3862 + 81(600) - 700$
701 to 800	1	800
801 to 900	81	$3862 + 81(800)$
901 to 1000	81	$3862 + 81(900)$
1001 to 1100	81	$3862 + 81(1000)$
1101 to 1200	81	$3862 + 81(1100)$
1201 to 1300	81	$3862 + 81(1200)$
1301 to 1400	81	$3862 + 81(1300)$
1401 to 1500	81	$3862 + 81(1400)$
1501 to 1600	81	$3862 + 81(1500)$
1601 to 1700	81	$3862 + 81(1600) - 1700$
1701 to 1800	1	1800
1801 to 1900	81	$3862 + 81(1800)$
1901 to 2000	81	$3862 + 81(1900)$

For the integers from 2001 to 2024, the only integers that contain the digit 7 are 2007 and 2017. Thus, the sum of the integers from 2001 to 2024 that do not contain a 7 is

$$24 \times 2000 + \frac{(24)(25)}{2} - 2007 - 2017 = 44\,276$$

Therefore, the overall sum of the integers from 1 to 2024 that do not contain the digit 7 is

$$18(3862) + 81(16\,600) - 700 + 800 - 1700 + 1800 + 44\,276 = 1\,458\,592$$

Solution 2

Consider first the integers from 000 to 999 that do not contain the digit 7. (We can include 000 in this list as it will not affect the sum.)

Since each of the three digits has 9 possible values (0, 1, 2, 3, 4, 5, 6, 8, or 9), there are $9 \times 9 \times 9 = 729$ such integers.

If we fix any specific digit in any of the three positions, there will be exactly $9 \times 9 = 81$ integers with that digit in that position, as there are 9 possibilities for each of the remaining digits. (For example, there are 81 such integers ending in 0, 81 ending in 1, etc.)

We sum these integers by first summing the units digit column, then summing the tens digit column, and then summing the hundreds digit column.

Since each of the 9 possible units digits occurs 81 times, the sum of the units digit column of all integers from 1 to 2024 that do not contain a 7 is

$$81(0) + 81(1) + 81(2) + 81(3) + 81(4) + 81(5) + 81(6) + 81(8) + 81(9) = 81(38)$$

Since each of the 9 possible tens digits occurs 81 times, the sum of the tens digit column of all integers from 1 to 2024 that do not contain a 7 is

$$81(0 + 10 + 20 + 30 + 40 + 50 + 60 + 80 + 90) = 81(380)$$

Similarly, the sum of the hundreds digits column is

$$81(3800)$$

Thus, the sum of the integers from 0 to 999 that do not contain the digit 7 is

$$81(38) + 81(380) + 81(3800) = 81(38)(1 + 10 + 100) = 81(38)(111) = 341\,658$$

Each of the 729 integers from 1000 to 1999 which do not contain 7 is 1000 more than such an integer between 0 and 999. There are again 729 of these integers, as the first digit is fixed at 1, and each of the remaining three digits has 9 possible values. Thus, the sum of these integers from 1000 to 1999 is equal to the sum of the corresponding ones from 0 to 999 plus 729(1000), or

$$341\,658 + 729\,000 = 1\,070\,658$$

For 2000 to 2024, the only integers that contain the digit 7 are 2007 and 2017. Thus, the sum of the integers from 2000 to 2024 that do not contain the digit 7 is

$$25 \times 2000 + \frac{(24)(25)}{2} - 2007 - 2017 = 46\,276$$

Therefore, the sum of the integers from 1 to 2024 that do not contain the digit 7 is

$$341\,658 + 1\,070\,658 + 46\,276 = 1\,458\,592$$

Problem of the Week

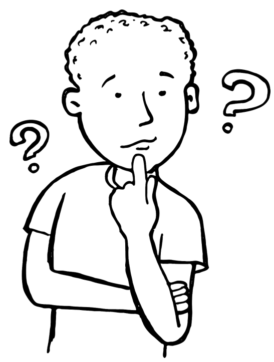
Problem E

The Choice is Yours

You are given the following list of the seven numbers: 10, 2, 5, 2, 4, 6, 2. You are asked to choose a positive number as the eighth number so that

- the mean, median, and mode of the eight numbers are distinct, and
- the mean, median, and mode differ by the same amount when put in order from least to greatest.

Determine all possible choices for the eighth number.



Problem of the Week
Problem E and Solution
The Choice is Yours

Problem

You are given the following list of the seven numbers: 10, 2, 5, 2, 4, 6, 2. You are asked to choose a positive number as the eighth number so that

- the mean, median, and mode of the eight numbers are distinct, and
- the mean, median, and mode differ by the same amount when put in order from least to greatest.

Determine all possible choices for the eighth number.

Solution

Let x represent the number chosen.

Since there are at least three 2s, the mode will be 2 regardless the value of x.

The mean of the numbers is $\frac{10+2+5+2+6+4+2+x}{8} = \frac{x+31}{8}$.

The median of the numbers will depend on the value of x compared to the other numbers. Since there are eight numbers in the list, the median will be the average of the fourth and fifth numbers in the list, ordered from least to greatest. We will break the problem into cases.

Case 1: $0 < x \leq 2$

Then the list, ordered from least to greatest, is x, 2, 2, 2, 4, 5, 6, 10.

The median is the average of the fourth and fifth numbers, or $\frac{2+4}{2} = 3$.

Since $x > 0$, the mean is $\frac{x+31}{8} > \frac{31}{8} > 3$. It follows that the mean is greater than the median. Then the mean, median, and mode in order from least to greatest is 2, 3, $\frac{x+31}{8}$.

Since the differences between adjacent numbers are equal, we have

$$\begin{aligned} 3-2 &= \frac{x+31}{8} - 3 \\ 4 &= \frac{x+31}{8} \\ 32 &= x+31 \\ 1 &= x \end{aligned}$$

When $x = 1$, the list of numbers, ordered from least to greatest, is 1, 2, 2, 2, 4, 5, 6, 10. The mean is 4, the median is 3, and the mode is 2. When listed from least to greatest, the three numbers are 2, 3, 4 and the difference between adjacent terms is 1. Therefore, $x = 1$ is a possible choice for the eighth number.

Case 2: $2 < x \leq 5$

Then the list, ordered from least to greatest, is 2, 2, 2, x, 4, 5, 6, 10 or 2, 2, 2, 4, x, 5, 6, 10.

In both lists, the fourth and fifth numbers are x and 4. It follows that the median for both lists is $\frac{x+4}{2}$. We also know that the mode is 2. However, we do not know which is larger, the median or the mean, so we examine both cases.

- **Case 2a:** The median is less than the mean.

  In this case, the mean, median, and mode, ordered from least to greatest, is 2, $\frac{x+4}{2}$, $\frac{x+31}{8}$. Since the differences between adjacent numbers are equal, we have
  $$\frac{x+4}{2} - 2 = \frac{x+31}{8} - \frac{x+4}{2}$$
  Multiplying both sides of the equation by 8,
  $$\begin{aligned} 4x + 16 - 16 &= x + 31 - 4x - 16 \\ 7x &= 15 \\ x &= \frac{15}{7} \end{aligned}$$
  When $x = \frac{15}{7}$, the list of numbers, ordered from least to greatest, is 2, 2, 2, $\frac{15}{7}$, 4, 5, 6, 10. The mean is $\frac{29}{7}$, the median is $\frac{43}{14}$, and the mode is 2. When listed from least to greatest, the three numbers are 2, $\frac{43}{14}$, $\frac{29}{7}$, and the difference between adjacent terms is $\frac{15}{14}$. Therefore, $x = \frac{15}{7}$ is a possible choice for the eighth number.

- **Case 2b:** The median is greater than the mean.

  In this case, the mean, median, and mode, ordered from least to greatest, is 2, $\frac{x+31}{8}$, $\frac{x+4}{2}$. Since the differences between adjacent numbers are equal, we have
  $$\frac{x+31}{8} - 2 = \frac{x+4}{2} - \frac{x+31}{8}$$
  Multiplying both sides of the equation by 8,
  $$\begin{aligned} x + 31 - 16 &= 4x + 16 - x - 31 \\ 30 &= 2x \\ x &= 15 \end{aligned}$$
  But $15 > 5$, so there is no value of x that satisfies the given conditions in this case.

Case 3: $x > 5$

Then the first five numbers in the list, in order from least to greatest, are 2, 2, 2, 4, 5.

Since the fourth number in the list is 4 and the fifth number is 5, then the median is $\frac{9}{2}$. The mode is 2. Since $x > 5$, the mean is $\frac{x+31}{8} > \frac{36}{8} = \frac{9}{2}$. It follows that the mean is greater than the median. The mean, median, and mode, ordered from least to greatest, is then 2, $\frac{9}{2}$, $\frac{x+31}{8}$.

Since the differences between adjacent numbers in the list are equal, we have
$$\begin{aligned} \frac{9}{2} - 2 &= \frac{x+31}{8} - \frac{9}{2} \\ 7 &= \frac{x+31}{8} \\ 56 &= x + 31 \\ 25 &= x \end{aligned}$$

When $x = 25$, the list of numbers, ordered from least to greatest, is 2, 2, 2, 4, 5, 6, 10, 25. The mean is 7, the median is $\frac{9}{2}$, and the mode is 2. When listed from least to greatest, the three numbers are 2, $\frac{9}{2}$, 7 and the difference between adjacent terms is $\frac{5}{2}$. Therefore, $x = 25$ is a possible choice for the eighth number.

Therefore, the possible choices for the eighth number are 1, $\frac{15}{7}$ and 25.

Problem of the Week

Problem E

Sums with Multiples of Three

The set $\{3, 6, 9, 12, 15, \ldots, 2022, 2025\}$ contains all of the multiples of three from 3 to 2025.

Three distinct numbers are chosen from the set to form a sum. How many different sums can be formed?

Problem of the Week
Problem E and Solution
Sums with Multiples of Three

Problem

The set $\{3, 6, 9, 12, 15, \ldots, 2022, 2025\}$ contains all of the multiples of three from 3 to 2025.

Three distinct numbers are chosen from the set to form a sum. How many different sums can be formed?

Solution

Since the set includes every multiple of three from 3 to 2025, then there are $2025 \div 3 = 675$ numbers in the set. Each number is of the form $3n$, for $n = 1, 2, 3, \ldots, 675$. The required sum is $3a + 3b + 3c$, where a, b, and c are three distinct numbers chosen from $\{1, 2, 3, \ldots, 675\}$. But $3a + 3b + 3c = 3(a + b + c)$. Thus, we can reduce the problem to the question of, "How many distinct integers can be formed by adding three numbers from the set $\{1, 2, 3, \ldots, 675\}$?"

The smallest sum is $1 + 2 + 3 = 6$ and the largest sum is $673 + 674 + 675 = 2022$. It is possible to get every number in between 6 and 2022 by:

(a) increasing the sum by 1 by replacing one of the three numbers with a number that is 1 larger or,

(b) decreasing the sum by 1 replacing one of the three numbers with a number that is 1 smaller.

Therefore, all of the integers from 6 to 2022 inclusive can be formed. There are 2022 integers from 1 to 2022, inclusive. But this includes the five integers from 1 to 5. So, there are $2022 - 5 = 2017$ integers from 6 to 2022. Thus, 2017 different sums can be formed.

This answer, 2017, is the answer to the original problem as well. If $a + b + c = 6$, then $3(a + b + c) = 18$. This is the smallest integer that is the sum of the three smallest numbers, 3, 6 and 9, from the original set. If $a + b + c = 2022$, then $3(a + b + c) = 6066$. This is the largest integer that is the sum of the three largest numbers, 2019, 2022, and 2025, from the original set. Then every multiple of three from 18 to 6066 can be generated by adding three different numbers from the original set. (There are 2017 multiples of three from 18 to 6066, inclusive. And each of these can be obtained by adding three distinct numbers from the original set.)

Problem of the Week

Problem E

A Tale of Two Towns

Two towns, Centreville and Middletown, had the same population at the end 2022.

The population of Centreville decreased by 2.5% from the end of 2022 to the end of 2023. Then, the population increased by 8.4% from the end of 2023 to the end of 2024.

The population of Middletown increased by $r\%$, where $r > 0$, from the end of 2022 to the end of 2023. Then, the population of Middletown increased by $(r + 2)\%$ from the end of 2023 to the end of 2024.

Surprisingly, the populations of both towns were the same again at the end of 2024. Determine the value of r, rounded to the nearest tenth.

Problem of the Week

Problem E and Solution

A Tale of Two Towns

Problem

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Surprisingly, the populations of both towns were the same again at the end of 2024. Determine the value of r, rounded to the nearest tenth.

Solution

Let p be the population of Centreville at the end of 2022. Since Centreville and Middletown have the same population size at the end of 2022, then p is also the population of Middletown at the end of 2022.

The population of Centreville decreased by 2.5% in 2023, so the population at the end of 2023 was

$$p - \frac{2.5}{100}p = \left(1 - \frac{2.5}{100}\right)p = 0.975p$$

The population of Centreville then increased by 8.4% during 2024, so the population at the end of 2024 was

$$0.975p + \left(\frac{8.4}{100}\right)(0.975p) = \left(1 + \frac{8.4}{100}\right)(0.975p) = 1.084(0.975p) = 1.0569p$$

The population of Middletown increased by $r\%$ in 2023, so the population at the end of 2023 was

$$p + \frac{r}{100}p = \left(1 + \frac{r}{100}\right)p$$

The population of Middletown then increased by $(r+2)\%$ during 2024, so the population at the end of 2024 was

$$\left(1 + \frac{r}{100}\right)p + \frac{r+2}{100}\left(1 + \frac{r}{100}\right)p = \left(1 + \frac{r}{100}\right)\left(1 + \frac{r+2}{100}\right)p$$

Since the populations of Centreville and Middletown are equal at the end of 2024, we have

$$\left(1 + \frac{r}{100}\right)\left(1 + \frac{r+2}{100}\right)p = 1.0569p$$

Dividing both sides by $p > 0$ and multiplying both sides by 10 000 to clear fractions, we have $(100 + r)(102 + r) = 10\,569$. Thus, $10\,200 + 202r + r^2 = 10\,569$, and so $r^2 + 202r - 369 = 0$.

Using the quadratic formula, $r = \dfrac{-202 \pm \sqrt{202^2 - 4(-369)}}{2} \approx 1.8, -203.8$.

Since $r > 0$, we have $r \approx 1.8\%$, correct to one decimal place.

Problem of the Week

Problem E

Sixty-Four!

The product $64 \times 63 \times 62 \times \cdots \times 3 \times 2 \times 1$ can be written as 64! and called "64 *factorial*".

In general, the product of the positive integers 1 to m is

$$m! = m \times (m-1) \times (m-2) \times \cdots \times 3 \times 2 \times 1$$

If 64! is divisible by 2025^n, determine the largest positive integer value of n.

64!

Problem of the Week
Problem E and Solution
Sixty-Four!

Problem

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In general, the product of the positive integers 1 to m is

$$m! = m \times (m-1) \times (m-2) \times \cdots \times 3 \times 2 \times 1$$

If 64! is divisible by 2025^n, determine the largest positive integer value of n.

Solution

Let $P = 64!$. The prime factorization of 2025 is $3^4 \times 5^2$. We must determine how many times the factors of 3 and 5 are repeated in the factorization of P.

First we count the number of factors of 3 in P by looking at the multiples of 3 from 1 to 64. They are 3, 6, 9, ..., 57, 60, and 63. Each of these 21 numbers contains a factor of 3.

Now, each multiple of 9 from 1 to 64 will contain a second factor of 3. These multiples of 9 are 9, 18, 27, 36, 45, 54, and 63. Each of these 7 numbers contains two factors of 3.

Now, each multiple of 27 from 1 to 64 will contain a third factor of 3. These multiples of 27 are 27 and 54. Each of these 2 numbers contains three factors of 3.

There are no higher powers of 3 less than 64. Thus, P has $21 + 7 + 2 = 30$ factors of 3, and so the largest power of 3 that P is divisible by is 3^{30}.

Next we count the number of factors of 5 in P by looking at the multiples of 5 from 1 to 64. They are 5, 10, 15, ..., 50, 55, and 60. Each of these 12 numbers contains a factor of 5.

Now, each multiple of 25 from 1 to 64 will contain a second factor of 5. These multiples of 25 are 25 and 50. Each of these 2 numbers contains two factors of 5.

There are no higher powers of 5 less than 64. Thus, P has $12 + 2 = 14$ factors of 5, and so the largest power of 5 that P is divisible by is 5^{14}.

Thus, P is divisible by $3^{30} \times 5^{14}$.

$$3^{30} \times 5^{14} = 3^{28} \times 3^2 \times 5^{14} = \left(3^4 \times 5^2\right)^7 \times 3^2 = 2025^7 \times 3^2$$

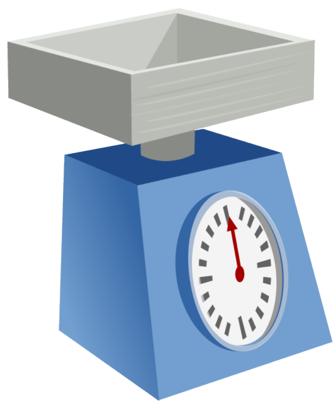
Thus, P is divisible by 2025^7, and 7 is the largest value of n.

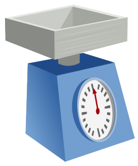
Problem of the Week

Problem E

A Heavy Problem

Four boxes: one blue, one green, one pink, and one yellow, each have a different mass. The mass of each box, in grams, is a positive integer. Inkeri does not know the individual masses of the boxes, but she knows the combined mass of the pink, blue, and green boxes is 13 grams. She also knows the combined mass of the blue, green, and yellow boxes is 17 grams, and the combined mass of the pink, green, and yellow boxes is 19 grams. Determine all possibilities for the individual masses of the boxes.





Problem of the Week

Problem E and Solution

A Heavy Problem

Problem

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Solution

Let b, g, p, and y represent the masses, in grams, of the blue, green, pink, and yellow boxes, respectively. It is given that $b \neq g \neq p \neq y$ and each of the masses is a positive integer. We also know the following:

$$p + b + g = 13 \tag{1}$$
$$b + g + y = 17 \tag{2}$$
$$p + g + y = 19 \tag{3}$$

Subtracting (2) − (1) gives $y - p = 4$, or equivalently, $y = p + 4$.

Subtracting (3) − (2) gives $p - b = 2$, or equivalently, $b = p - 2$.

From here we proceed with two different approaches.

Solution 1

From (1), we obtain $g = 13 - p - b$. Since p, b, g, and y are all different positive integers, we can look at the possible values of p and calculate the values of b, g, and y in each case. Then we can determine if it is a possible set of masses. This is summarized in the following table.

Mass of Pink Box, p	Mass of Blue Box, $b = p - 2$	Mass of Green Box, $g = 13 - p - b$	Mass of Yellow Box, $y = p + 4$	Possible?
1	-1	13	5	No, $b < 1$
2	0	11	6	No, $b < 1$
3	1	9	7	Yes
4	2	7	8	Yes
5	3	5	9	No, $p = g$
6	4	3	10	Yes
7	5	1	11	Yes

We can stop here, because if $p > 7$, then $g < 1$, which is not valid. Therefore, there are four different possibilities for the masses of the blue, green, pink, and yellow boxes, respectively. They are:

$$(b, g, p, y) = (1, 9, 3, 7),\ (2, 7, 4, 8),\ (4, 3, 6, 10),\ (5, 1, 7, 11)$$

Solution 2

This solution is similar to Solution 1. The key difference is that in this solution, we find expressions for b, g, and y, in terms of p. We then determine the smallest and largest possible values for p.

We substitute $y = p + 4$ and $b = p - 2$ into (2) to get an expression for g in terms of p.

$$\begin{aligned} b + g + y &= 17 \\ (p-2) + g + (p+4) &= 17 \\ 2p + g + 2 &= 17 \\ g &= 15 - 2p \end{aligned}$$

Since $b = p - 2$ and b is a positive integer, the smallest positive integer value for p will be 3. Otherwise $b < 1$. Since $g = 15 - 2p$ and g is a positive integer, the largest positive integer value for p will be 7. Otherwise $g < 1$. Therefore, the only possible values for p are 3, 4, 5, 6, and 7.

We will now look at each possible value of p, calculate the values of b, g, and y in each case, and determine if it is a possible set of masses.

Mass of Pink Box, p	Mass of Blue Box, $b = p - 2$	Mass of Green Box, $g = 15 - 2p$	Mass of Yellow Box, $y = p + 4$	Possible?
3	1	9	7	Yes
4	2	7	8	Yes
5	3	5	9	No, $p = g$
6	4	3	10	Yes
7	5	1	11	Yes

Therefore, there are four different possibilities for the masses of the blue, green, pink, and yellow boxes, respectively. They are:

$$(b, g, p, y) = (1, 9, 3, 7),\ (2, 7, 4, 8),\ (4, 3, 6, 10),\ (5, 1, 7, 11)$$

Problem of the Week

Problem E

There and Back

Silvan and Nash are training for a race. They run 600 m along a straight path from their school to a bridge. Once they reach the bridge they turn around and run along the same path back to their school. Silvan and Nash run at different, but constant speeds for each part of the run. For each of them, their constant speed from the bridge back to the school was twice their constant speed on the way to the bridge.

Silvan reached the bridge first, turned around, and immediately started running back to school. Silvan then passed Nash running in the opposite direction when they were 50 m away from the bridge.

When Silvan reaches the school, how far behind him will Nash be?

Problem of the Week
Problem E and Solution
There and Back

Problem

Silvan and Nash are training for a race. They run 600 m along a straight path from their school to a bridge. Once they reach the bridge they turn around and run along the same path back to their school. Silvan and Nash run at different, but constant speeds for each part of the run. For each of them, their constant speed from the bridge back to the school was twice their constant speed on the way to the bridge.

Silvan reached the bridge first, turned around, and immediately started running back to school. Silvan then passed Nash running in the opposite direction when they were 50 m away from the bridge.

When Silvan reaches the school, how far behind him will Nash be?

Solution

Let s represent Silvan's speed, in metres per second, when running to the bridge, and $2s$ represent his speed when running from the bridge back to school. Let n represent Nash's speed, in metres per second, when running to the bridge, and $2n$ represent his speed when running from the bridge back to school.

When Silvan and Nash passed each other 50 m away from the bridge, they had been running for the same amount of time. In this time, Silvan had run 600 m to the bridge at a speed of s m/s and 50 m back to school at a speed of $2s$ m/s, and Nash had run 550 m to the bridge at a speed of n m/s. Thus,

$$\begin{aligned} \frac{600}{s} + \frac{50}{2s} &= \frac{550}{n} \\ \frac{600}{s} + \frac{25}{s} &= \frac{550}{n} \\ \frac{625}{s} &= \frac{550}{n} \\ \frac{n}{s} &= \frac{550}{625} = \frac{22}{25} \end{aligned}$$

Let x represent the distance, in metres, that Nash is behind Silvan when Silvan reaches the school. At this time, Silvan will have run 600 m to the bridge at a speed of s m/s and 600 m back to school at a speed of $2s$ m/s, and Nash will have run 600 m to the bridge at a speed of n m/s and $(600 - x)$ m back to school at a speed of $2n$ m/s. Thus,

$$\begin{aligned}\frac{600}{s}+\frac{600}{2s}&=\frac{600}{n}+\frac{600-x}{2n}\\ \frac{600}{s}+\frac{300}{s}&=\frac{1200}{2n}+\frac{600-x}{2n}\\ \frac{900}{s}&=\frac{1800-x}{2n}\\ \frac{2n}{s}&=\frac{1800-x}{900}\\ \frac{n}{s}&=\frac{1800-x}{1800}\end{aligned}$$

From earlier, we know that $\frac{n}{s}=\frac{22}{25}$. Thus,

$$\begin{aligned}\frac{22}{25}&=\frac{1800-x}{1800}\\ 22(1800)&=25(1800-x)\\ 39\,600&=45\,000-25x\\ 25x&=5400\\ x&=216\end{aligned}$$

Therefore, when Silvan reaches the school, Nash will be 216 m behind him.

Problem of the Week

Problem E

1225 is Even More Special

Did you know that 1225 can be written as the sum of seven consecutive integers?

That is,

$$1225 = 172 + 173 + 174 + 175 + 176 + 177 + 178$$

The notation below illustrates a mathematical short form used for writing the above sum. This notation is called *Sigma Notation*.

$$\sum_{i=172}^{178} i = 1225$$

How many ways can the number 1225 be expressed as the sum of an **even** number of consecutive positive integers?

$$\sum_{i=118}^{127} i = 1225$$

Problem of the Week
Problem E and Solution
1225 is Even More Special

Problem

Did you know that 1225 can be written as the sum of seven consecutive integers?

That is,

$$1225 = 172 + 173 + 174 + 175 + 176 + 177 + 178$$

How many ways can the number 1225 be expressed as the sum of an **even** number of consecutive positive integers?

Solution

Suppose k is even. We can write k consecutive integers as

$$n - \left(\frac{k}{2} - 1\right),\ \ldots,\ n,\ n+1,\ \ldots,\ n + \left(\frac{k}{2} - 1\right),\ n + \frac{k}{2}$$

Here, n and $n+1$ are the middle numbers in the sum, and there are $\frac{k}{2} - 1$ integers less than n in the sum and $\frac{k}{2}$ integers greater than n in the sum.

We can write the sum of these integers in this way:

$$\left(n - \left(\frac{k}{2} - 1\right)\right) + \cdots + n + (n+1) + \cdots + \left(n + \left(\frac{k}{2} - 1\right)\right) + \left(n + \frac{k}{2}\right)$$

This simplifies to $kn + \frac{k}{2}$.

For example, four consecutive integers can be expressed as $n-1$, n, $n+1$, and $n+2$, where n is an integer.
Their sum is $(n-1) + n + (n+1) + (n+2) = 4n + 2$.

Notice that $kn + \frac{k}{2} = k(n + \frac{1}{2})$. Thus, if this sum is equal to 1225, then $k(n + \frac{1}{2}) = 1225$. Multiplying both sides by 2, we have

$$2k\left(n + \frac{1}{2}\right) = 2(1225)$$
$$k(2n+1) = 2450$$

Since n is an integer, then $2n+1$ is an odd integer. Therefore, we're looking for factor pairs of 2450, where one factor is even and the other is odd.

Since $2450 = 2(5^2)(7^2)$, the positive odd divisors of 2450 are 1, 5, 7, 25, 35, 49, 175, 245 and 1225.

For each odd divisor, $2n+1$, of 2450, we determine n and $k = \frac{2450}{2n+1}$. The k integers that sum to 1225 will then be $n - \left(\frac{k}{2} - 1\right), \ldots, n, n+1, \ldots, n + \left(\frac{k}{2} - 1\right), n + \frac{k}{2}$. This is summarized in the table below.

Odd Divisor $(2n+1)$	n	**Number of integers** (k)	**Sum of Integers**
1	0	2450	$(-1224)+(-1223)+\cdots+0+\cdots+1224+1225$
5	2	490	$(-242)+(-241)+\cdots+2+\cdots+246+247$
7	3	350	$(-171)+(-170)+\cdots+3+\cdots+177+178$
25	12	98	$(-36)+(-35)+\cdots+12+\cdots+60+61$
35	17	70	$(-17)+(-16)+\cdots+17+\cdots+51+52$
49	24	50	$0+1+\cdots+24+\cdots+48+49$
175	87	14	$81+82+83+84+85+86+87+88+89+90+91+92+93+94$
245	122	10	$118+119+120+121+122+123+124+125+126+127$
1225	612	2	$612+613$

For $k = 14$, 10, and 2, all integers in the sum are positive.

Thus, there are three ways to express 1225 as the sum of an even number of consecutive positive integers.

Extension: Determine the number of ways the number 1225 can be expressed as the sum of an **odd** number of consecutive positive integers.

Problem of the Week

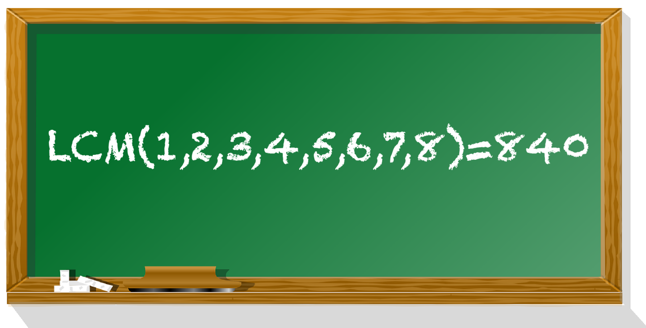
Problem E

Four More

For each positive integer n, $\text{LCM}(1, 2, \ldots, n)$ is the *least common multiple* of $1, 2, \ldots, n$. That is, the smallest positive integer divisible by each of $1, 2, \ldots, n$.

Determine all positive integers n, with $1 \leq n \leq 100$ such that

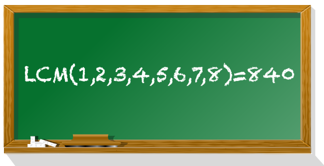
$$\text{LCM}(1, 2, \ldots, n) = \text{LCM}(1, 2, \ldots, n + 4)$$



NOTE: In solving this problem, it might be helpful to know that we can calculate the LCM of a set of positive integers by

- determining the prime factorization of each integer in the set,
- determining the list of prime numbers that occur in these prime factorizations,
- determining the highest power of each prime number from this list that occurs in the prime factorizations, and
- multiplying these highest powers together.

For example, $\text{LCM}(1, 2, 3, 4, 5, 6, 7, 8) = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 840$, since the prime factorizations of 2, 3, 4, 5, 6, 7, and 8 are 2, 3, 2^2, 5, $2 \cdot 3$, 7, and 2^3, respectively.



Problem of the Week

Problem E and Solution

Four More

Problem

For each positive integer n, $\text{LCM}(1, 2, \ldots, n)$ is the *least common multiple* of $1, 2, \ldots, n$. That is, the smallest positive integer divisible by each of $1, 2, \ldots, n$.

Determine all positive integers n, with $1 \le n \le 100$ such that

$$\text{LCM}(1, 2, \ldots, n) = \text{LCM}(1, 2, \ldots, n + 4)$$

NOTE: In solving this problem, it might be helpful to know that we can calculate the LCM of a set of positive integers by

- determining the prime factorization of each integer in the set,
- determining the list of prime numbers that occur in these prime factorizations,
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Solution

Since $\text{LCM}(1, 2, \ldots, n)$ is the least common multiple of $1, 2, \ldots, n$ and $\text{LCM}(1, 2, \ldots, n + 4)$ is the least common multiple of $1, 2, \ldots, n, n + 1, n + 2, n + 3, n + 4$, then $\text{LCM}(1, 2, \ldots, n) \neq \text{LCM}(1, 2, \ldots, n + 4)$ if either

(i) there are prime factors that occur in $n + 1$, $n + 2$, $n + 3$, $n + 4$ that don't occur in any of $1, 2, \ldots, n$, or

(ii) there is a higher power of a prime that occurs in the factorizations of one of $n + 1$, $n + 2$, $n + 3$, $n + 4$ that doesn't occur in any of $1, 2, \ldots, n$.

For (i) to occur, consider a prime p that is a divisor of one of $n + 1$, $n + 2$, $n + 3$, $n + 4$, and none of $1, 2, \ldots, n$.. This means that the smallest positive integer that has p as a divisor is one of the integers $n + 1$, $n + 2$, $n + 3$, $n + 4$, which in fact means that this integer equals p. (The smallest multiple of a prime p is $1 \cdot p = p$ itself.)

Thus, for (i) to occur, one of $n+1$, $n+2$, $n+3$, $n+4$ is a prime number.

For (ii) to occur, consider a prime power p^k, where k is an integer > 1, that is a divisor of one of $n+1$, $n+2$, $n+3$, $n+4$ and none of $1, 2, \ldots, n$. This means that the smallest positive integer that has p^k as a divisor is one of the integers $n+1$, $n+2$, $n+3$, $n+4$, which in fact means that this integer equals p^k. (The smallest multiple of p^k is p^k itself.)

Therefore, $\text{LCM}(1, 2, \ldots, n) \neq \text{LCM}(1, 2, \ldots, n+4)$ whenever one of $n+1$, $n+2$, $n+3$, $n+4$ is a prime number or a prime power.

In other words, $\text{LCM}(1, 2, \ldots, n) = \text{LCM}(1, 2, \ldots, n+4)$ whenever none of $n+1$, $n+2$, $n+3$, $n+4$ is a prime number or a prime power.

Therefore, we want to determine the positive integers n with $1 \leq n \leq 100$ for which none of $n+1$, $n+2$, $n+3$, $n+4$ is a prime number or a prime power.

The prime numbers less than or equal to 104 are

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97,\ 101,\ 103$$

The prime powers, with exponent > 1, less than or equal to 104 are

$$2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 5^2 = 25,\ 7^2 = 49$$

Therefore, we want to count the positive integers n with $1 \leq n \leq 100$ for which none of $n+1$, $n+2$, $n+3$, $n+4$ appear in the list

$$2,\ 3,\ 4,\ 5,\ 7,\ 8,\ 9,\ 11,\ 13,\ 16,\ 17,\ 19,\ 23,\ 25,\ 27,\ 29,\ 31,\ 32,\ 37,\ 41,\ 43,\ 47,\ 49,\ 53,\ 59,\ 61,\ 64,\ 67,\ 71,\ 73,\ 79,\ 81,\ 83,\ 89,\ 97,\ 101,\ 103$$

For four consecutive integers to not occur in this list, we need a difference between adjacent numbers to be at least 5.

The values of n that satisfy this condition are $n = 32$, 53, 54, 73, 74, 83, 84, 89, 90, 91, 92. (For example, 54 is a value of n that works since none of 55, 56, 57, 58 appears in the list.)

Therefore, there are 11 values of n with $1 \leq n \leq 100$ for which $\text{LCM}(1, 2, \ldots, n) = \text{LCM}(1, 2, \ldots, n+4)$.

Problem of the Week

Problem E

Mystery Function

For some function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are integers, we know the following information:

- the y-intercept is 5,
- $f(2) = -3$,
- $f(4)$ is greater than 40 but less than 50, and
- $f(6)$ is greater than 240 but less than 250.

Determine the value of $f(7)$.

Problem of the Week
Problem E and Solution
Mystery Function

Problem

For some function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are integers, we know the following information:

- the y-intercept is 5,
- $f(2) = -3$,
- $f(4)$ is greater than 40 but less than 50, and
- $f(6)$ is greater than 240 but less than 250.

Determine the value of $f(7)$.

Solution

Since the y-intercept is 5, it follows that $f(0) = 5$. Thus,

$$\begin{aligned} a(0)^3 + b(0)^2 + c(0) + d &= 5 \\ d &= 5 \end{aligned}$$

We can now write the function as $f(x) = ax^3 + bx^2 + cx + 5$.

Since $f(2) = -3$,

$$\begin{aligned} a(2)^3 + b(2)^2 + c(2) + 5 &= -3 \\ 8a + 4b + 2c + 5 &= -3 \\ 8a + 4b + 2c &= -8 \\ 4a + 2b + c &= -4 \\ c &= -4a - 2b - 4 \qquad (1) \end{aligned}$$

Next we consider $f(4)$.

$$\begin{aligned} f(4) &= a(4)^3 + b(4)^2 + c(4) + 5 \\ &= 64a + 16b + 4c + 5 \\ &= 64a + 16b + 4(-4a - 2b - 4) + 5 \qquad \text{(using equation (1))} \\ &= 64a + 16b - 16a - 8b - 16 + 5 \\ &= 48a + 8b - 11 \end{aligned}$$

Since $f(4) > 40$, it follows that $48a + 8b - 11 > 40$ or $48a + 8b > 51$. Dividing the inequality by 8 gives $6a + b > 6.375$. Similarly, since $f(4) < 50$, it follows that $48a + 8b - 11 < 50$ or $48a + 8b < 61$. Dividing the inequality by 8 gives $6a + b < 7.625$. Since a and b are integers it follows that $6a + b$ is an integer. Thus, since $6a + b > 6.375$ and $6a + b < 7.625$, we can conclude that $6a + b = 7$.

Next we consider $f(6)$.

$$\begin{aligned} f(6) &= a(6)^3 + b(6)^2 + c(6) + 5 \\ &= 216a + 36b + 6c + 5 \\ &= 216a + 36b + 6(-4a - 2b - 4) + 5 \qquad \text{(using equation (1))} \\ &= 216a + 36b - 24a - 12b - 24 + 5 \\ &= 192a + 24b - 19 \end{aligned}$$

Since $f(6) > 240$, it follows that $192a + 24b - 19 > 240$, or $192a + 24b > 259$. Dividing the inequality by 24 gives $8a + b > 10\frac{19}{24}$. Similarly, since $f(6) < 250$, it follows that $192a + 24b - 19 < 250$, or $192a + 24b < 269$. Dividing the inequality by 24 gives $8a + b < 11\frac{5}{24}$. Since a and b are integers it follows that $8a + b$ is an integer. Thus, since $8a + b > 10\frac{19}{24}$ and $8a + b < 11\frac{5}{24}$, we can conclude that $8a + b = 11$.

We now have the following system of equations.

$$6a + b = 7 \qquad (2)$$

$$8a + b = 11 \qquad (3)$$

By subtracting equation (2) from equation (3), we obtain $2a = 4$, or $a = 2$. Substituting $a = 2$ in equation (2) gives $6(2) + b = 7$, and thus $b = -5$.

Substituting $a = 2$ and $b = -5$ in equation (1) gives:

$$\begin{aligned} c &= -4a - 2b - 4 \\ &= -4(2) - 2(-5) - 4 \\ &= -8 + 10 - 4 = -2 \end{aligned}$$

We can now write the function as $f(x) = 2x^3 - 5x^2 - 2x + 5$.

Finally we can determine $f(7)$.

$$\begin{aligned} f(7) &= 2(7)^3 - 5(7)^2 - 2(7) + 5 \\ &= 2(343) - 5(49) - 14 + 5 \\ &= 686 - 245 - 9 = 432 \end{aligned}$$

Therefore, $f(7) = 432$.

NOTE:

We could have written the third bullet point as $40 < f(4) < 50$ and solved the entire inequality at once instead of dealing with the inequality symbols one at a time. While this may be unfamiliar to students, it's a helpful way to solve inequalities. This would have looked as follows.

$$\begin{aligned} 40 &< f(4) &&< 50 \\ 40 &< 48a + 8b - 11 &&< 50 \\ 51 &< 48a + 8b &&< 61 \\ 6.375 &< 6a + b &&< 7.625 \end{aligned}$$

From here, we can conclude that since $6a + b$ is an integer, then we must have $6a + b = 7$. We could have then used a similar approach to solve the inequalities in $f(6)$.