Problem of the Week
Problem E and Solution
The Six Dollar Game

Problem

A two-player game has players alternating turns removing coins from three piles. At the beginning of the game, the first pile contains 1 coin, the second pile contains 2 coins, and the third pile contains 3 coins. Players take turns removing one or more coins from any **one** pile. On their turn, a player can remove from one coin to all of the coins in a particular pile. The player who removes the last coin wins the game.

Jessie and Jason play this game. Jessie goes first. Jason claims that no matter what Jessie does on his turn, that Jason can win by following his winning strategy.

Describe Jason's winning strategy.

Solution

There are six possibilities for Jessie's first move: he can remove 1 coin from the first, second, or third pile, or 2 coins from the second or third pile, or all 3 coins from the third pile. In each case, Jason can always remove coins in some way so that when his first turn is over there are two piles with the same number of coins and one pile with no coins remaining, as follows.

- **Case 1:** Jessie removes 1 coin from the first pile.

  Jason should remove 1 coin from the third pile on his first turn. The result will be 0 coins in the first pile, 2 coins in the second pile, and 2 coins in the third pile.

- **Case 2:** Jessie removes 1 coin from the second pile.

  Jason should remove all 3 coins from the third pile on his first turn. The result will be 1 coin in the first pile, 1 coin in the second pile, and 0 coins in the third pile.

- **Case 3:** Jessie removes 1 coin from the third pile.

  Jason should remove 1 coin from the first pile on his first turn. The result will be 0 coins in the first pile, 2 coins in the second pile, and 2 coins in the third pile.

- **Case 4:** Jessie removes 2 coins from the second pile.

  Jason should remove 2 coins from the third pile on his first turn. The result will be 1 coin in the first pile, 0 coins in the second pile, and 1 coin in the third pile.

- **Case 5:** Jessie removes 2 coins from the third pile.

  Jason should remove 2 coins from the second pile on his first turn. The result will be 1 coin in the first pile, 0 coins in the second pile, and 1 coin in the third pile.

- **Case 6:** Jessie removes 3 coins from the third pile.

  Jason should remove 1 coin from the second pile on his first turn. The result will be 1 coin in the first pile, 1 coin in the second pile, and 0 coins in the third pile.

In each of the six cases, Jason leaves Jessie with only two piles, each containing the same number of coins. More specifically, after Jason's first turn there will be either two piles with 2 coins in each or two piles with 1 coin in each.

If the two remaining piles each contain 1 coin, Jessie must then remove 1 coin from one of the piles on his second turn. Then on his second turn Jason removes the final coin and wins.

If the two remaining piles each contain 2 coins, Jessie has two choices for his second turn: Jessie can either remove 2 coins from one of the piles or 1 coin from one of the piles.

- If Jessie removes 2 coins from one of the piles, Jason then removes 2 coins from the remaining pile and wins.

- If Jessie removes 1 coin from one of the piles on his second turn, then Jason responds by removing 1 coin from the other pile. After Jason's second turn there would be two piles remaining, each containing 1 coin. Jessie must then remove 1 coin from one of the piles on his third turn. Jason then removes the final coin on his third turn and wins.

In summary, Jason's strategy is as follows. After Jessie's first turn, Jason removes coin(s) from a pile so that only two piles remain and each pile contains the same number of coins. On his second and possible third turn, he copies what Jessie did on his previous turn, but to the pile he did not touch.

For Further Thought:

Jessie proposes two new versions of the coin game. For these new versions, does either player have a winning strategy?

(a) Suppose there are just two piles to start, one pile has 13 coins and the other has 17 coins. On any turn a player can remove any number of coins from one of the piles. Players alternate turns and the player to remove the last coin wins.

(b) Suppose there are three piles of coins to start, one pile has 2 coins, a second pile has 4 coins, and the third pile has 5 coins. On any turn a player can remove any number of coins from one of the piles. Players alternate turns and the player to remove the last coin wins.