Problem of the Week
Problem E and Solution
Forgotten Code

Problem

Suk-Ja has a combination lock that contains 12 wheels, each with the letters in her name. Thus, the code to open the lock is 12 characters long and contains only the letters S, U, K, J, and A. Unfortunately, Suk-Ja has forgotten the code. She remembers the following information and plans to use this to determine all the possible codes.

- There were exactly three As in the code, located in positions 3, 6, and 12, when numbered from left to right.
- The letter S was always followed by either U or K.
- The letter U was always followed by either K or J.
- The letter K was always followed by either J or A.
- The letter J was always followed by either A or S.
- The letter A was always followed by either S or U.

For example, one possible code is $UJAUKASKJSKA$.

How many different possible codes are there?

Solution

To determine the total number of possible codes, we first look at how many ways we can fill positions 1 and 2, then positions 4 and 5, and finally positions 7 to 11.

When filling positions 1 and 2, we note position 3 must be A. Working backwards, we know that position 2 must then be either K or J.

- If position 2 is K, then position 1 can be either S or U.
- If position 2 is J, then position 1 can be either U or K.

Thus, there are 4 ways to fill positions 1 and 2, namely SK, UK, UJ, or KJ.

When filling positions 4 and 5, we note that positions 3 and 6 are both A. Working backwards from position 6, we know that position 5 must then be either K or J. Working forwards from position 3, we know that position 4 must then be either S or U.

- If position 4 is S, then position 5 can be either U or K. However we already determined that position 5 must be either K or J, so position 5 cannot be U.

- If position 4 is U, then position 5 can be either K or J.

Thus, there are 3 ways to fill positions 4 and 5, namely SK, UK, or UJ.

When filling positions 7 to 11, we note that A cannot be in any of these positions, since the only As are in positions 3, 6, and 12. Since A is in position 12, we know that position 11 must be either K or J. Similarly, since A is in position 6, we know that position 7 must then be either S or U.

- If position 7 is S, then position 8 can be either U or K. Thus, positions 7 and 8 can be SU or SK. From here, positions 7, 8, and 9 can be SUK, SUJ, or SKJ. We cannot have SKA since A cannot be in positions 7 to 11. From here, positions 7 to 10 can be $SUKJ$, $SUJS$, or $SKJS$. From here, positions 7 to 11 can be $SUKJS$, $SUJSU$, $SUJSK$, $SKJSU$, or $SKJSK$. However, we already determined that position 11 must be either K or J. Thus, the only two possibilities for positions 7 to 11 with S in position 7 are $SUJSK$ or $SKJSK$.
- If position 7 is U, then position 8 can be either K or J. Thus, positions 7 and 8 can be UK or UJ. From here, positions 7, 8, and 9 can be UKJ or UJS. We cannot have UKA or UJA since A cannot be in positions 7 to 11. From here, positions 7 to 10 can be $UKJS$, $UJSU$, or $UJSK$. From here, positions 7 to 11 can be $UKJSU$, $UKJSK$, $UJSUK$, $UJSUJ$, or $UJSKJ$. However, we already determined that position 11 must be either K or J. Thus, the only four possibilities for positions 7 to 11 with U in position 7 are $UKJSK$, $UJSUK$, $UJSUJ$, or $UJSKJ$.

Thus, there are $2 + 4 = 6$ ways to fill positions 7 to 11.

There are 4 ways to fill positions 1 and 2. For each of these, there are 3 ways to fill positions 4 and 5. Thus, there are $4 \times 3 = 12$ ways to fill positions 1, 2, 4, and 5. For each of these, there are 6 ways to fill positions 7 to 11. Thus, there are $12 \times 6 = 72$ different possible codes.