Problem of the Week
Problem E and Solution
Mystery Function

Problem

For some function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are integers, we know the following information:

- the y-intercept is 5,
- $f(2) = -3$,
- $f(4)$ is greater than 40 but less than 50, and
- $f(6)$ is greater than 240 but less than 250.

Determine the value of $f(7)$.

Solution

Since the y-intercept is 5, it follows that $f(0) = 5$. Thus,

$$\begin{aligned} a(0)^3 + b(0)^2 + c(0) + d &= 5 \\ d &= 5 \end{aligned}$$

We can now write the function as $f(x) = ax^3 + bx^2 + cx + 5$.

Since $f(2) = -3$,

$$\begin{aligned} a(2)^3 + b(2)^2 + c(2) + 5 &= -3 \\ 8a + 4b + 2c + 5 &= -3 \\ 8a + 4b + 2c &= -8 \\ 4a + 2b + c &= -4 \\ c &= -4a - 2b - 4 \qquad (1) \end{aligned}$$

Next we consider $f(4)$.

$$\begin{aligned} f(4) &= a(4)^3 + b(4)^2 + c(4) + 5 \\ &= 64a + 16b + 4c + 5 \\ &= 64a + 16b + 4(-4a - 2b - 4) + 5 \qquad \text{(using equation (1))} \\ &= 64a + 16b - 16a - 8b - 16 + 5 \\ &= 48a + 8b - 11 \end{aligned}$$

Since $f(4) > 40$, it follows that $48a + 8b - 11 > 40$ or $48a + 8b > 51$. Dividing the inequality by 8 gives $6a + b > 6.375$. Similarly, since $f(4) < 50$, it follows that $48a + 8b - 11 < 50$ or $48a + 8b < 61$. Dividing the inequality by 8 gives $6a + b < 7.625$. Since a and b are integers it follows that $6a + b$ is an integer. Thus, since $6a + b > 6.375$ and $6a + b < 7.625$, we can conclude that $6a + b = 7$.

Next we consider $f(6)$.

$$\begin{aligned} f(6) &= a(6)^3 + b(6)^2 + c(6) + 5 \\ &= 216a + 36b + 6c + 5 \\ &= 216a + 36b + 6(-4a - 2b - 4) + 5 \qquad \text{(using equation (1))} \\ &= 216a + 36b - 24a - 12b - 24 + 5 \\ &= 192a + 24b - 19 \end{aligned}$$

Since $f(6) > 240$, it follows that $192a + 24b - 19 > 240$, or $192a + 24b > 259$. Dividing the inequality by 24 gives $8a + b > 10\frac{19}{24}$. Similarly, since $f(6) < 250$, it follows that $192a + 24b - 19 < 250$, or $192a + 24b < 269$. Dividing the inequality by 24 gives $8a + b < 11\frac{5}{24}$. Since a and b are integers it follows that $8a + b$ is an integer. Thus, since $8a + b > 10\frac{19}{24}$ and $8a + b < 11\frac{5}{24}$, we can conclude that $8a + b = 11$.

We now have the following system of equations.

$$6a + b = 7 \qquad (2)$$

$$8a + b = 11 \qquad (3)$$

By subtracting equation (2) from equation (3), we obtain $2a = 4$, or $a = 2$. Substituting $a = 2$ in equation (2) gives $6(2) + b = 7$, and thus $b = -5$.

Substituting $a = 2$ and $b = -5$ in equation (1) gives:

$$\begin{aligned} c &= -4a - 2b - 4 \\ &= -4(2) - 2(-5) - 4 \\ &= -8 + 10 - 4 = -2 \end{aligned}$$

We can now write the function as $f(x) = 2x^3 - 5x^2 - 2x + 5$.

Finally we can determine $f(7)$.

$$\begin{aligned} f(7) &= 2(7)^3 - 5(7)^2 - 2(7) + 5 \\ &= 2(343) - 5(49) - 14 + 5 \\ &= 686 - 245 - 9 = 432 \end{aligned}$$

Therefore, $f(7) = 432$.

NOTE:

We could have written the third bullet point as $40 < f(4) < 50$ and solved the entire inequality at once instead of dealing with the inequality symbols one at a time. While this may be unfamiliar to students, it's a helpful way to solve inequalities. This would have looked as follows.

$$\begin{aligned} 40 < f(4) &< 50 \\ 40 < 48a + 8b - 11 &< 50 \\ 51 < 48a + 8b &< 61 \\ 6.375 < 6a + b &< 7.625 \end{aligned}$$

From here, we can conclude that since $6a + b$ is an integer, then we must have $6a + b = 7$. We could have then used a similar approach to solve the inequalities in $f(6)$.