$$\sum_{i=118}^{127} i = 1225$$

Problem of the Week
Problem E and Solution
1225 is Even More Special

Problem

Did you know that 1225 can be written as the sum of seven consecutive integers?

That is,

$$1225 = 172 + 173 + 174 + 175 + 176 + 177 + 178$$

How many ways can the number 1225 be expressed as the sum of an **even** number of consecutive positive integers?

Solution

Suppose k is even. We can write k consecutive integers as

$$n - \left(\frac{k}{2} - 1\right),\ \ldots,\ n,\ n+1,\ \ldots,\ n + \left(\frac{k}{2} - 1\right),\ n + \frac{k}{2}$$

Here, n and $n+1$ are the middle numbers in the sum, and there are $\frac{k}{2} - 1$ integers less than n in the sum and $\frac{k}{2}$ integers greater than n in the sum.

We can write the sum of these integers in this way:

$$\left(n - \left(\frac{k}{2} - 1\right)\right) + \cdots + n + (n+1) + \cdots + \left(n + \left(\frac{k}{2} - 1\right)\right) + \left(n + \frac{k}{2}\right)$$

This simplifies to $kn + \frac{k}{2}$.

For example, four consecutive integers can be expressed as $n-1$, n, $n+1$, and $n+2$, where n is an integer.
Their sum is $(n-1) + n + (n+1) + (n+2) = 4n + 2$.

Notice that $kn + \frac{k}{2} = k(n + \frac{1}{2})$. Thus, if this sum is equal to 1225, then $k(n + \frac{1}{2}) = 1225$. Multiplying both sides by 2, we have

$$2k\left(n + \frac{1}{2}\right) = 2(1225)$$
$$k(2n+1) = 2450$$

Since n is an integer, then $2n + 1$ is an odd integer. Therefore, we're looking for factor pairs of 2450, where one factor is even and the other is odd.

Since $2450 = 2(5^2)(7^2)$, the positive odd divisors of 2450 are 1, 5, 7, 25, 35, 49, 175, 245 and 1225.

For each odd divisor, $2n+1$, of 2450, we determine n and $k = \frac{2450}{2n+1}$. The k integers that sum to 1225 will then be $n - \left(\frac{k}{2} - 1\right)$, …, n, $n+1$, …, $n + \left(\frac{k}{2} - 1\right)$, $n + \frac{k}{2}$. This is summarized in the table below.

Odd Divisor $(2n+1)$	n	**Number of integers** (k)	**Sum of Integers**
1	0	2450	$(-1224)+(-1223)+\cdots+0+\cdots+1224+1225$
5	2	490	$(-242)+(-241)+\cdots+2+\cdots+246+247$
7	3	350	$(-171)+(-170)+\cdots+3+\cdots+177+178$
25	12	98	$(-36)+(-35)+\cdots+12+\cdots+60+61$
35	17	70	$(-17)+(-16)+\cdots+17+\cdots+51+52$
49	24	50	$0+1+\cdots+24+\cdots+48+49$
175	87	14	$81+82+83+84+85+86+87+88+89+90+91+92+93+94$
245	122	10	$118+119+120+121+122+123+124+125+126+127$
1225	612	2	$612+613$

For $k = 14$, 10, and 2, all integers in the sum are positive.

Thus, there are three ways to express 1225 as the sum of an even number of consecutive positive integers.

Extension: Determine the number of ways the number 1225 can be expressed as the sum of an **odd** number of consecutive positive integers.