Problem of the Week
Problem D and Solution
Squares in a Square

Problem

The prime factorization of 20 is $2^2 \times 5$.

The number 20 has 6 positive divisors. They are:

$$2^0 5^0 = 1,\ 2^0 5^1 = 5,\ 2^1 5^0 = 2,\ 2^1 5^1 = 10,\ 2^2 5^0 = 4,\ \ 2^2 5^1 = 20$$

Two of the divisors, 1 and 4, are perfect squares.

How many positive divisors of 36^3 are perfect squares?

Solution

First, let's look at the prime factorization of four different perfect squares:

$$9 = 3^2,\ 16 = 2^4,\ 36 = 2^2 \times 3^2,\ \ 129\,600 = 2^6 \times 3^4 \times 5^2$$

Note that, in each case, the exponent on each of the prime factors is even. In fact, a positive integer is a perfect square exactly when the exponent on each of the prime factors in its prime factorization is an even integer greater than or equal to zero. Now

$$\begin{aligned} 36^3 &= (2^2 \times 3^2)^3 \\ &= (2^2)^3 \times (3^2)^3 \\ &= 2^6 \times 3^6 \end{aligned}$$

All positive divisors of 36^3 will be of the form $2^k \times 3^n$ where k and n are integers with $0 \leq k \leq 6$ and $0 \leq n \leq 6$.

For $2^k \times 3^n$ to be a perfect square, k and n must be even integers. Thus, $k \in \{0, 2, 4, 6\}$ and $n \in \{0, 2, 4, 6\}$.

For each of the 4 values of k, there are 4 values of n, so there are $4 \times 4 = 16$ perfect square divisors of 36^3.

Therefore, 36^3 has 16 divisors that are perfect squares.

We can systematically list all of the divisors that are perfect squares. They are:

$2^0 3^0 = 1$	$2^0 3^2 = 9$	$2^0 3^4 = 81$	$2^0 3^6 = 729$
$2^2 3^0 = 4$	$2^2 3^2 = 36$	$2^2 3^4 = 324$	$2^2 3^6 = 2916$
$2^4 3^0 = 16$	$2^4 3^2 = 144$	$2^4 3^4 = 1296$	$2^4 3^6 = 11\,664$
$2^6 3^0 = 64$	$2^6 3^2 = 576$	$2^6 3^4 = 5184$	$2^6 3^6 = 46\,656$