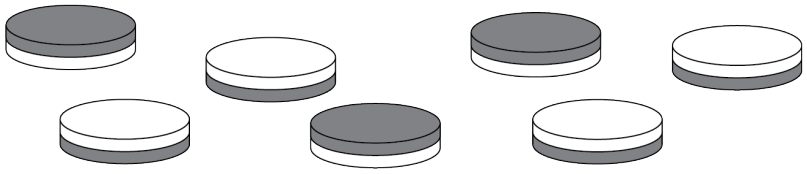
Problem of the Week
Problem D and Solution
Flipping Coins

Problem

Yousaf has several plastic coins that are white on one side and grey on the other. He places 16 of these coins on a table with their white sides facing up, and flips them according to the following rules:

1. Flipping a coin changes the colour of the side facing up from either white to grey or from grey to white.
2. Yousaf flips exactly 3 different coins at a time. Each time he does this it is called a *step*.

Determine the fewest steps required until all 16 coins have their grey sides facing up.



Extension:
Suppose Yousaf now starts with 2026 coins on the table with their white sides all facing up. Determine the fewest steps required until all 2026 coins have their grey sides facing up.

Solution

To simplify the language, a *grey coin* is a coin with its grey side facing up, and a *white coin* is a coin with its white side facing up. We start with 16 white coins and after some number of steps, we want to have 16 grey coins. There are four possible options for each step, assuming there are enough white or grey coins on the table. These options are listed in the following table, along with how each step affects the total number of grey coins:

Step Description	Change in Number of Grey Coins
flip 3 white coins to grey	increase by 3
flip 2 white coins to grey and 1 grey coin to white	increase by 1
flip 1 white coin to grey and 2 grey coins to white	decrease by 1
flip 3 grey coins to white	decrease by 3

Since we want the fewest steps possible, we should aim to flip 3 white coins to grey in as many of the steps as we can, as that will increase the total number of

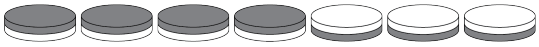
grey coins the fastest. If the total number of coins on the table were a multiple of 3, then in every step we could flip 3 white coins to grey. However 16 is not a multiple of 3. Therefore, in at least one of our steps we must *not* flip 3 white coins to grey.

To start with, in each of the first 3 steps we will flip 3 white coins to grey. There will then be $3 \times 3 = 9$ grey coins and $16 - 9 = 7$ white coins.

From here, we will only consider the remaining 7 white coins.

In step 4, we flip 3 white coins to grey. This results in 3 grey coins and 4 white coins, as shown.

In step 5, we flip 2 white coins to grey and 1 grey coin to white. This results in 4 grey coins and 3 white coins, as shown.



Finally, in step 6, we flip 3 white coins to grey. This results in 7 grey coins, as shown.

This method took a total of 6 steps to flip all 16 coins from white to grey. Since only one of these steps did *not* flip 3 white coins to grey, and we determined that at least one of the steps must do this, we could not have achieved our goal in fewer steps.

Therefore, the fewest steps until all 16 coins have their grey sides facing up is 6.

Solution to Extension:

We can use a similar approach for 2026 coins. We notice that $2026 = 3 \times 673 + 7$. Thus, if the first 673 steps are to flip 3 white coins to grey, then we will be left with 7 white coins. We already determined that the fewest steps required to flip 7 white coins to grey is 3. Thus, the fewest steps required until all 2026 coins have their grey sides facing up is $673 + 3 = 676$.