$2

Problem of the Week

Problem D and Solution

The Twos and Fives Dilemma

$5

Problem

Virat has a large collection of \$2 bills and \$5 bills. He makes stacks of bills that each have a total value of exactly \$100. Each stack has a least one \$2 bill, at least one \$5 bill, and no other types of bills. If each stack has a different number of \$2 bills than any other stack, what is the maximum number of stacks that Virat can create?

Solution

Consider a stack of bills with a total value of \$100 that includes x \$2 bills and y \$5 bills. The \$2 bills are worth $\$2x$ and the \$5 bills are worth $\$5y$, and so $2x + 5y = 100$.

Determining the number of possible stacks that the teller could have is equivalent to determining the numbers of pairs (x, y) of integers with $x \geq 1$ and $y \geq 1$ and $2x + 5y = 100$ or $5y = 100 - 2x$. (We must have $x \geq 1$ and $y \geq 1$ because each stack includes at least one \$2 bill and at least one \$5 bill.)

Since $x \geq 1$, then

$$\begin{aligned} 2x &\geq 2 \\ -2x &\leq -2 \\ 100 - 2x &\leq 100 - 2 \\ 100 - 2x &\leq 98 \end{aligned}$$

Also, since $5y = 100 - 2x$, this becomes $5y \leq 98$.

This means that $y \leq \frac{98}{5} = 19.6$. Since y is an integer, then $y \leq 19$.

Notice that since $5y = 100 - 2x$, then the right side is the difference between two even integers and is therefore even. This means that $5y$ (the left side) is even, which means that y must be even.

Since y is even, $y \geq 1$, and $y \leq 19$, then the possible values of y are 2, 4, 6, 8, 10, 12, 14, 16, and 18.

Each of these values gives a pair (x, y) that satisfies the equation $2x + 5y = 100$. These ordered pairs are $(45, 2)$, $(40, 4)$, $(35, 6)$, $(30, 8)$, $(25, 10)$, $(20, 12)$, $(15, 14)$, $(10, 16)$, and $(5, 18)$.

Therefore, the maximum number of stacks that Virat can create is 9.