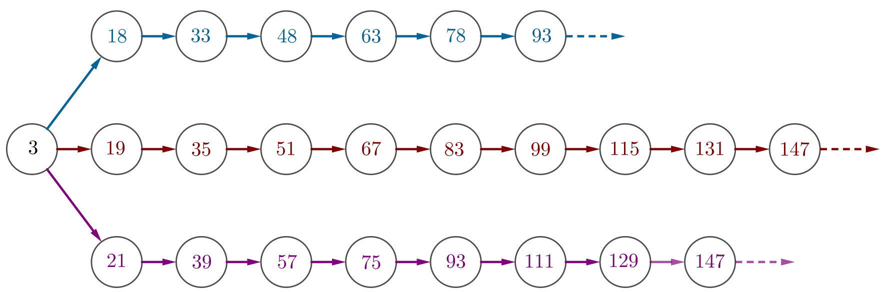
Problem of the Week

Problem D and Solution

Three Sequences

Problem

Three sequences each have first term 3. In the first sequence, each term after the first term is 15 more than the previous term. In the second sequence, each term after the first term is 16 more than the previous term. In the third sequence, each term after the first term is 18 more than the previous term. The three sequences continue indefinitely.



Determine all numbers between 3 and 2025, inclusive, that are common to all three sequences.

Solution

Solution 1

One could write out each sequence to a term less than or equal to 2025. You would write 135 terms of the first sequence, 127 terms of the second sequence and 113 terms of the third sequence. At this point you would compare the three sequences to find three numbers, 3, 723, and 1443, that are common to all three sequences. This is not a practical solution if you are solving the problem using pencil and paper. However, a solver could write a computer program that would easily handle this problem.

Solution 2

Notice that the number 147 occurs in both the second and third sequences. This number is $147 - 3 = 144$ greater than the first number common to both sequences. What is the significance of 144? It is the Least Common Multiple (LCM) of 16 and 18. The number 16 written in terms of prime factors is $2 \times 2 \times 2 \times 2 = 2^4$ and the number 18 written in terms of prime factors is $2 \times 3 \times 3 = 2 \times 3^2$. To find the LCM of 16 and 18, we determine the highest power that appears on each prime number in the two prime factorizations, and multiply all primes to the highest power together. Since the highest power of 2 in the two factorizations is 2^4 and the highest power of 3 in the two factorizations is 3^2, we have that the LCM of 16 and 18 is $2^4 \times 3^2 = 144$.

Now, we can create a fourth sequence that starts with 3, and each term after the first term is 144 greater than the previous term. The first few terms of this sequence are 3, 147, 291, 435, 579, 723, 867.

What numbers are common to this fourth sequence and the first sequence? We need to find the LCM of 15, the amount that each term in the first sequence increases by, and 144, the amount that each term in the fourth sequence increases by. The number 15 written in terms of prime factors is 3×5 and the number 144 written in terms of prime factors is $2^4 \times 3^2$. Therefore, the LCM of 15 and 144 is $2^4 \times 3^2 \times 5 = 720$.

Now, we create a fifth sequence that starts with 3, and each term after the first term is 720 greater than the previous term. This sequence contains all numbers that would be common to each of the three given sequences. The first few terms would be 3, 723, 1443, 2163.

Therefore, there are three numbers between 3 and 2025, inclusive, common to all three sequences. These numbers are 3, 723, and 1443.

Solution 3

This solution builds on the ideas in Solution 2. To solve the problem, we need to find the Least Common Multiple, or LCM, of 15, 16, and 18. First, we write each of the three numbers as a product of their prime factors.

$$\begin{aligned}15 &= 3 \times 5\\16 &= 2 \times 2 \times 2 \times 2 = 2^4\\18 &= 2 \times 3 \times 3 = 2 \times 3^2\end{aligned}$$

The LCM of 15, 16 and 18 is equal to the product of the highest power that appears on each prime number in the three prime factorizations. Since the highest power of 2 in the three factorizations is 2^4, the highest power of 3 is 3^2, and the highest power of 5 is 5^1, we have that the LCM of 15, 16, and 18 is $2^4 \times 3^2 \times 5 = 720$.

We can now determine the numbers between 3 and 2025 that would be common to all three sequences. The numbers are 3, $3 + 720 = 723$ and $723 + 720 = 1443$. If we were to continue, the next common number would be $1443 + 720 = 2163$, which is greater than 2025.

Therefore, there are three numbers between 3 and 2025, inclusive, common to all three sequences. These numbers are 3, 723, and 1443.