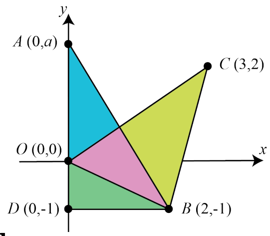



# Problem of the Week

## Problem D and Solution

## Stained Glass

### Problem

Points $A(0,a)$, $B(2,-1)$, $C(3,2)$, $D(0,-1)$, and $O(0,0)$ are such that $\triangle ABD$ and $\triangle COB$ have the same area. If $a>0$, determine the value of $a$.

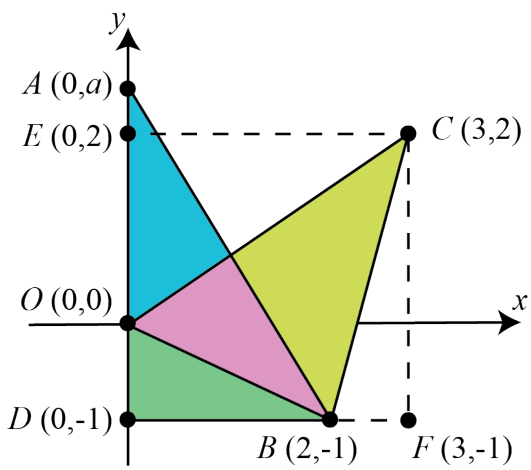
## Solution

### Solution 1

In $\triangle ABD$, $AD = a-(-1) = a+1$ and $DB = 2-0 = 2$.

Thus, area $\triangle ABD = \frac{AD\times DB}{2} = \frac{(a+1)\times 2}{2} = a+1$ units$^2$.

To determine the area of $\triangle COB$, we consider points $E(0,2)$ and $F(3,-1)$ and draw in $ECFD$.




Since $E$ and $D$ both have $x$-coordinate 0, $ED$ is a vertical line which passes through $O$. Since $C$ and $F$ have the same $x$-coordinate, $CF$ is also a vertical line. Since $E$ and $C$ have the same $y$-coordinate, $EC$ is a horizontal line. Since $D$ and $F$ both have $y$-coordinate $-1$, $DF$ is also a horizontal line which passes through $B$. Thus, $ECFD$ is a rectangle that encloses $\triangle COB$, and we have

$$\text{area } \triangle COB = \text{area } ECFD - \text{area } \triangle CEO - \text{area } \triangle ODB - \text{area } \triangle BFC$$

In rectangle $ECFD$, $EC = 3-0 = 3$ and $ED = 2-(-1) = 3$. The area of rectangle $EDFC = EC\times ED = 3\times 3 = 9$ units$^2$.

Since $ECFD$ is a rectangle, $\triangle CEO$ is right-angled at $E$. Since $EC = 3$ and $EO = 2-0 = 2$, the area of $\triangle CEO = \frac{EC\times EO}{2} = \frac{3\times 2}{2} = 3$ units$^2$.

Since $ECFD$ is a rectangle, $\triangle ODB$ is right-angled at $D$. Since $OD = 0-(-1) = 1$ and $DB = 2-0 = 2$, the area of $\triangle ODB = \frac{OD\times DB}{2} = \frac{1\times 2}{2} = 1$ unit$^2$.

Since $ECFD$ is a rectangle, $\triangle BFC$ is right-angled at $F$. Since $BF = 3-2 = 1$ and $CF = 2-(-1) = 3$, the area of $\triangle BFC = \frac{BF\times CF}{2} = \frac{1\times 3}{2} = 1.5$ units$^2$.

Thus,
$$\begin{aligned}\text{area } \triangle COB &= \text{area } ECFD - \text{area } \triangle CEO - \text{area } \triangle ODB - \text{area } \triangle BFC \\ &= 9 - 3 - 1 - 1.5 \\ &= 3.5 \text{ units}^2\end{aligned}$$

We're given that $\triangle ABD$ and $\triangle COB$ have the same area. Thus, the area of $\triangle ABD = 3.5$ units$^2$.

Since the area of $\triangle ABD = a + 1$ units$^2$, we have $a + 1 = 3.5$ and $a = 2.5$ follows.

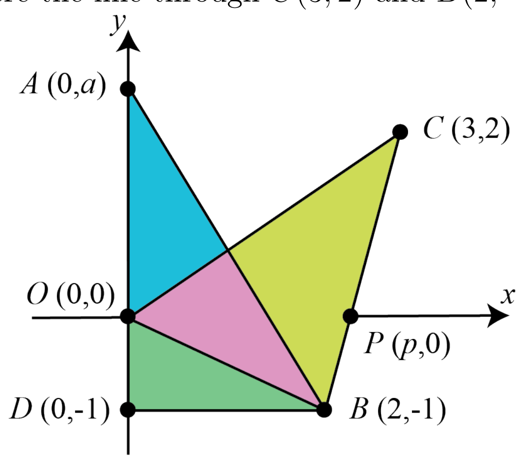
Therefore, the value of $a$ is 2.5.

**Solution 2**

In $\triangle ABD$, $AD = a - (-1) = a + 1$ and $DB = 2 - 0 = 2$.

Thus, area $\triangle ABD = \frac{AD \times DB}{2} = \frac{(a+1)\times 2}{2} = a + 1$ units$^2$.

Let $P(p, 0)$ be the point where the line through $C(3, 2)$ and $B(2, -1)$ intersects the $x$-axis.




We have area $\triangle COB$ = area $\triangle COP$ + area $\triangle BOP$.

To determine the value of $p$, we first determine the equation of the line through $C(3, 2)$ and $B(2, -1)$.

Since the slope of the line is $\frac{2-(-1)}{3-2} = 3$, the equation of the line is of the form $y = 3x + b$, for some $b$. Substituting $x = 3$ and $y = 2$ gives $2 = 3(3) + b$ and $b = -7$ follows. Therefore, the equation of the line though $C(3, 2)$ and $B(2, -1)$ is $y = 3x - 7$.

Substituting $x = p$ and $y = 0$ into $y = 3x - 7$ we obtain $0 = 3p - 7$ and $p = \frac{7}{3}$ follows.

In $\triangle COP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the $x$-axis to $C(3, 2)$, which is 2 units. Therefore, the area of $\triangle COP = \frac{\frac{7}{3}\times 2}{2} = \frac{7}{3}$ units$^2$.

In $\triangle BOP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the $x$-axis to $B(2, -1)$, which is 1 unit. Therefore, the area of $\triangle BOP = \frac{\frac{7}{3}\times 1}{2} = \frac{7}{6}$ units$^2$.

Therefore, area $\triangle COB$ = area $\triangle COP$ + area $\triangle BOP = \frac{7}{3} + \frac{7}{6} = \frac{14}{6} + \frac{7}{6} = \frac{21}{6} = \frac{7}{2}$ units$^2$.

We're given that $\triangle ABD$ and $\triangle COB$ have the same area. Thus, the area of $\triangle ABD = \frac{7}{2}$ units$^2$.

Since the area of $\triangle ABD = a + 1$ units$^2$, we have $a + 1 = \frac{7}{2}$ and $a = \frac{5}{2} = 2.5$ follows.

Therefore, the value of $a$ is 2.5.