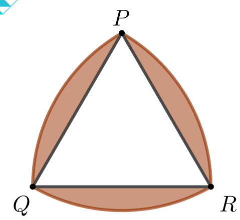


Problem of the Week

Problem D and Solution

Painting a Logo

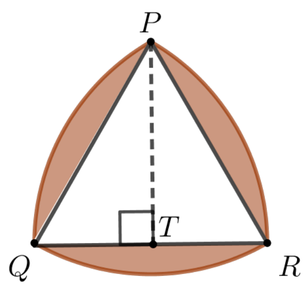
Problem

Nathaniel has designed a new logo for his school's math club. When drawing his logo, he starts with an equilateral triangle, labelled $\triangle PQR$, with sides of length 20 cm. He then draws in minor arc PQ, which is an arc of the circle with centre R and radius RQ, followed by minor arc PR, which is an arc of the circle with centre Q and radius QP, and then minor arc RQ, which is an arc of a circle with centre P and radius PR.

He wants to colour the region bounded by each arc but outside of $\triangle PQR$. Determine the total area to be coloured, correct to one decimal place.

Solution

We'll first determine the area of $\triangle PQR$. Construct altitude PT. Since $\triangle PQR$ is equilateral, it follows that PT bisects QR.



Since $QR = 20$ cm, it follows that $TR = 10$ cm. By the Pythagorean Theorem, $PR^2 = TR^2 + PT^2$. Therefore, $20^2 = 10^2 + PT^2$, and $PT^2 = 400 - 100 = 300$ follows. Since $PT > 0$, we have $PT = \sqrt{300}$ cm.

Therefore, the area of $\triangle PQR$ is $\dfrac{(QR) \times (PT)}{2} = \dfrac{20 \times \sqrt{300}}{2} = 10\sqrt{300}$ cm^2.

The logo consists of three overlapping circle sectors, one with centre P, one with centre Q, and one with centre R. Each circle sector has the same radius, 20 cm, and a 60° central angle. Therefore, each sector has the same area, which is $60 \div 360$, or one-sixth, the area of a circle of radius 20 cm.

That is, the area of each sector is equal to $\frac{1}{6}\pi r^2 = \frac{1}{6}\pi(20)^2 = \frac{200}{3}\pi$ cm^2.

The coloured part of each circle sector is equal to the area of the sector minus the area of $\triangle PQR$. That is, it is equal to

$$\left(\frac{200}{3}\pi - 10\sqrt{300}\right)\ \text{cm}^2$$

Since there are three congruent coloured areas, the total area to be coloured is equal to

$$3 \times \left(\frac{200}{3}\pi - 10\sqrt{300}\right) = 200\pi - 30\sqrt{300} \approx 108.7\ \text{cm}^2$$