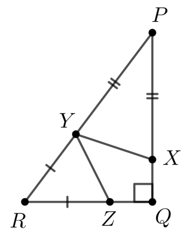


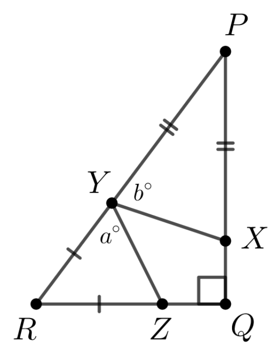
Problem of the Week
Problem D and Solution
Isosceles Delight

Problem

$\triangle PQR$ is right-angled at Q. Point X lies on PQ, point Z lies on QR, and point Y lies on PR such that $PX = PY$ and $RZ = RY$.

Determine the measure of $\angle XYZ$.

Solution
Solution 1



Let $\angle RYZ = a^\circ$ and $\angle PYX = b^\circ$.
Since $RY = RZ$, $\triangle RYZ$ is isosceles and $\angle RZY = \angle RYZ = a^\circ$.
Since $PY = PX$, $\triangle PYX$ is isosceles and $\angle PXY = \angle PYX = b^\circ$.

The angles in a triangle sum to 180°. Therefore, in $\triangle RYZ$,

$$\begin{aligned}\angle YRZ + \angle RYZ + \angle RZY &= 180^\circ\\ \angle YRZ + a^\circ + a^\circ &= 180^\circ\\ \angle YRZ &= 180^\circ - 2a^\circ\end{aligned}$$

In $\triangle PQR$,

$$\begin{aligned}\angle RPQ + \angle PQR + \angle QRP &= 180^\circ\\ \angle RPQ + 90^\circ + (180^\circ - 2a^\circ) &= 180^\circ\\ \angle RPQ &= 2a^\circ - 90^\circ\end{aligned}$$

In $\triangle PYX$,

$$\begin{aligned}\angle PYX + \angle PXY + \angle YPX &= 180^\circ\\ b^\circ + b^\circ + \angle YPX &= 180^\circ\end{aligned}$$

Since $\angle YPX = \angle RPQ$ (same angle), we have

$$\begin{aligned}b^\circ + b^\circ + (2a^\circ - 90^\circ) &= 180^\circ\\ 2b^\circ &= 270^\circ - 2a^\circ\\ b^\circ &= 135^\circ - a^\circ\end{aligned}$$

Now, PYR forms a straight line, so $\angle PYR = 180^\circ$. That is,

$$\begin{aligned}\angle PYX + \angle XYZ + \angle RYZ &= 180^\circ\\ b^\circ + \angle XYZ + a^\circ &= 180^\circ\\ (135^\circ - a^\circ) + \angle XYZ + a^\circ &= 180^\circ\\ \angle XYZ &= 180^\circ - 135^\circ = 45^\circ\end{aligned}$$

Therefore, $\angle XYZ = 45°$. Note that in solving for $\angle XYZ$ it is was not necessary to determine either $a°$ or $b°$.

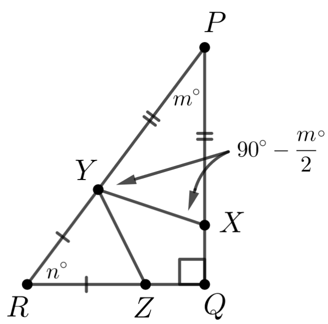
Solution 2

In $\triangle PQR$, let $\angle RPQ = m°$ and $\angle PRQ = n°$. The angles in a triangle sum to $180°$. Therefore, in $\triangle PQR$,

$$\begin{aligned} \angle RPQ + \angle PQR + \angle PRQ &= 180° \\ m° + 90° + n° &= 180° \\ m° + n° &= 90° \end{aligned}$$

Therefore, $m + n = 90$.

Since $PY = PX$, $\triangle PYX$ is isosceles and so $\angle PXY = \angle PYX$.
Also, in $\triangle PYX$



$$\begin{aligned} \angle PYX + \angle PXY + \angle YPX &= 180° \\ \angle PYX + \angle PYX + m° &= 180° \\ 2\angle PYX &= 180° - m° \end{aligned}$$

Therefore, $\angle PXY = \angle PYX = 90° - \left(\frac{m}{2}\right)°$.

Similarly, since $RY = RZ$, $\triangle RYZ$ is isosceles, and therefore $\angle RYZ = \angle RZY$. Also, in $\triangle RYZ$,

$$\begin{aligned} \angle RYZ + \angle RZY + \angle YRZ &= 180° \\ \angle RYZ + \angle RYZ + n° &= 180° \\ 2\angle RYZ &= 180° - n° \end{aligned}$$

Therefore, $\angle RYZ = \angle RZY = 90° - \left(\frac{n}{2}\right)°$.

Since $\angle PYR = 180°$, we have

$$\begin{aligned} \angle PYX + \angle XYZ + \angle RYZ &= 180° \\ \left(90° - \left(\frac{m}{2}\right)°\right) + \angle XYZ + \left(90° - \left(\frac{n}{2}\right)°\right) &= 180° \\ 180° - \left(\frac{m}{2}\right)° - \left(\frac{n}{2}\right)° + \angle XYZ &= 180° \\ \angle XYZ &= \left(\frac{m}{2}\right)° + \left(\frac{n}{2}\right)° \\ &= \left(\frac{m+n}{2}\right)° \end{aligned}$$

Therefore, since $m + n = 90$, we have $\angle XYZ = \left(\frac{90}{2}\right)° = 45°$.

Therefore, $\angle XYZ = 45°$. Note that in solving for $\angle XYZ$ it is was not necessary to determine either $m°$ or $n°$.