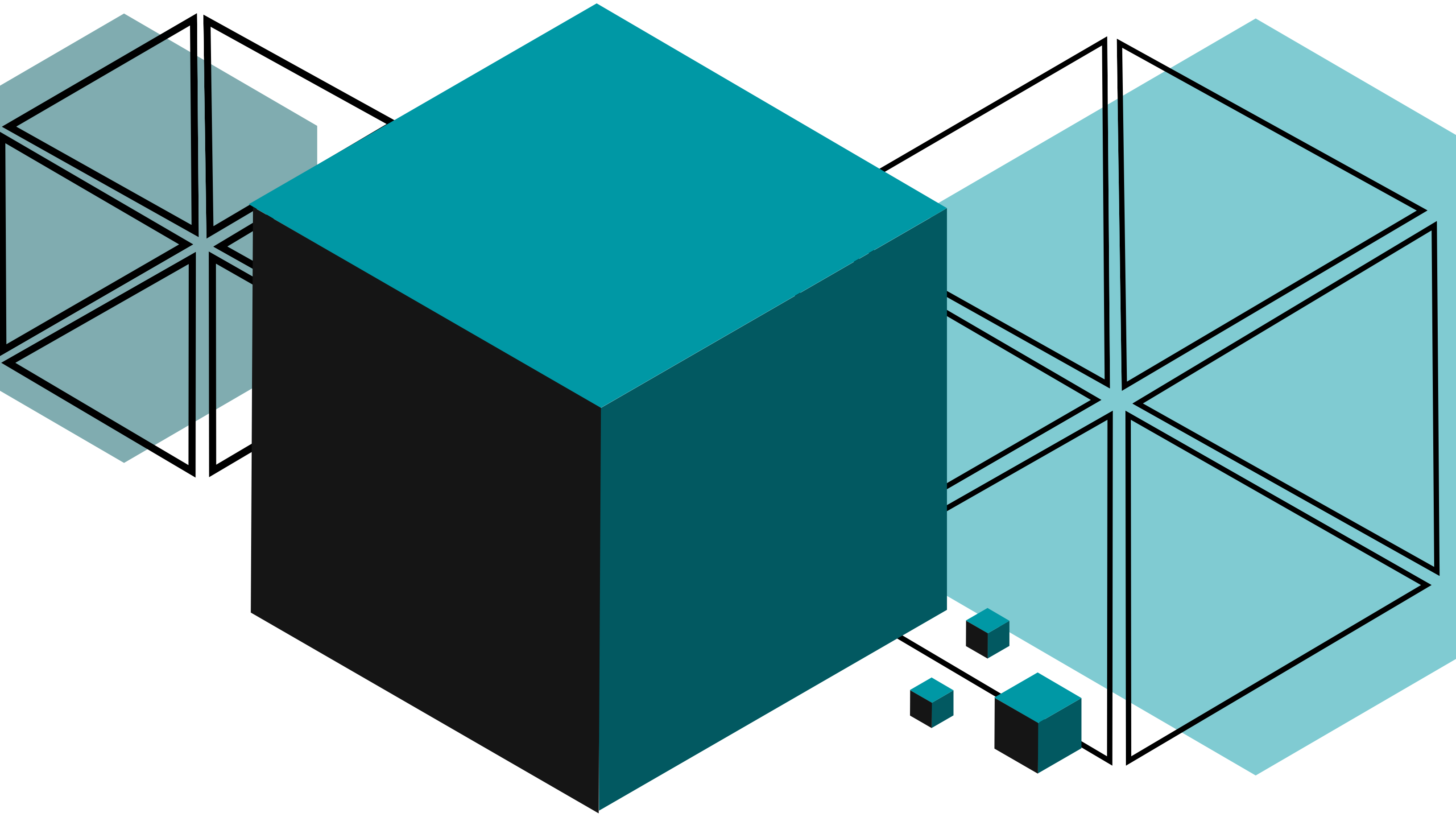
Problem of the Week

Problems and Solutions 2024-2025



Problem D

Grade 9/10

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

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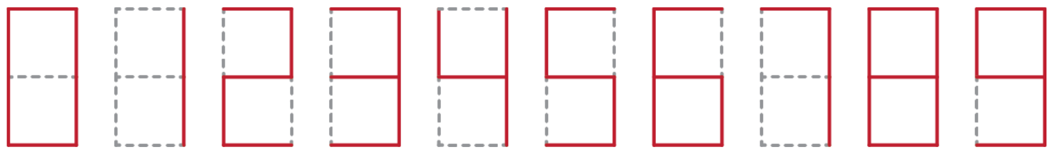
Problem of the Week

Problem D

The Clock Works

Halina's clock uses a digital LED display where each digit is represented by seven LED segments that are either on or off, as shown.



Sometimes some of the LED segments stop working. When the top-most horizontal LED segment stopped working, both the digit 1 and the digit 7 appeared as shown. This was a problem because Halina couldn't distinguish between them.

Halina replaced the broken LED segment, but then a week later found that a different LED segment had stopped working. However, this time, she was still able to distinguish between all ten digits.

What is the largest number of LED segments that can be broken at the same time, while still allowing Halina to distinguish between all ten digits?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Problem of the Week

Problem D and Solution

The Clock Works

Problem

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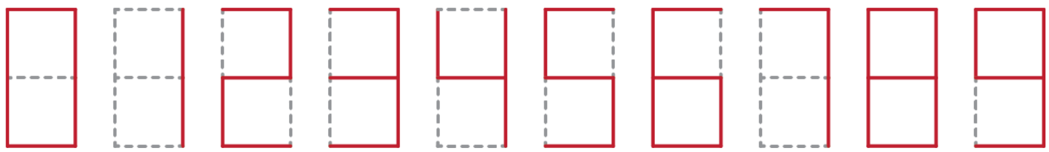

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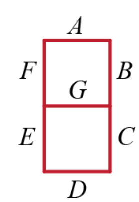
What is the largest number of LED segments that can be broken at the same time, while still allowing Halina to distinguish between all ten digits?

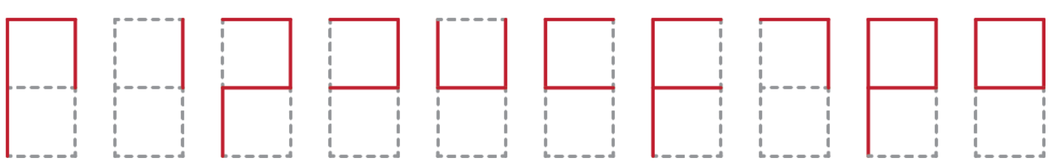
Solution

First we will label the seven LED segments as shown.



If segment A is broken, Halina cannot distinguish between digits 1 and 7.
If segment B is broken, Halina cannot distinguish between digits 6 and 8.
If segment E is broken, Halina cannot distinguish between digits 5 and 6, and also between digits 8 and 9.
If segment F is broken, Halina cannot distinguish between digits 3 and 9.
If segment G is broken, Halina cannot distinguish between digits 0 and 8.

If both segments C and D are broken, the ten digits would appear as shown.



Since each of these digits are unique, Halina can distinguish between them. Therefore, at most 2 LED segments can be broken at the same time.

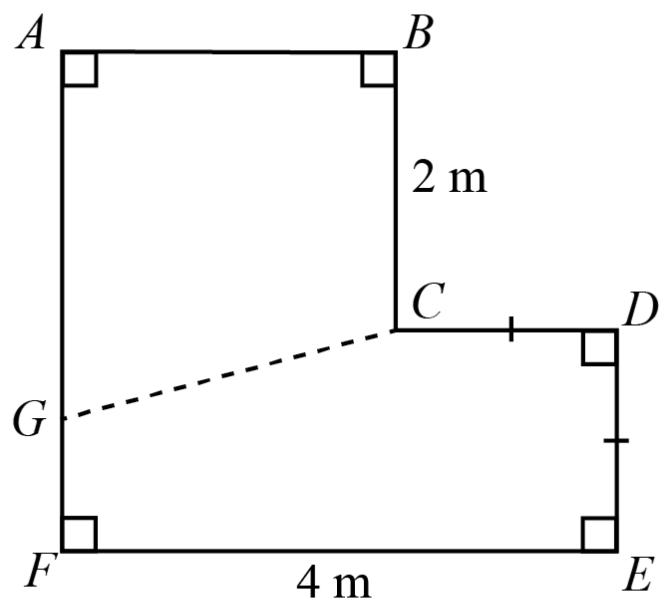
Problem of the Week

Problem D

Dividing Line

The Bobsie twins share an L-shaped room. The area of the entire room is 11.2 m^2. The twins are not getting along, so their parents decide to partition the room with tape so that each child has exactly the same area.

The layout of their room is represented by $ABCDEF$ in the diagram. The partitioning tape, indicated by a dashed line, will travel from C to a point G on AF.



Where should G be located on AF in order to split the room into two smaller rooms of equal area?

Problem of the Week
Problem D and Solution
Dividing Line

Problem

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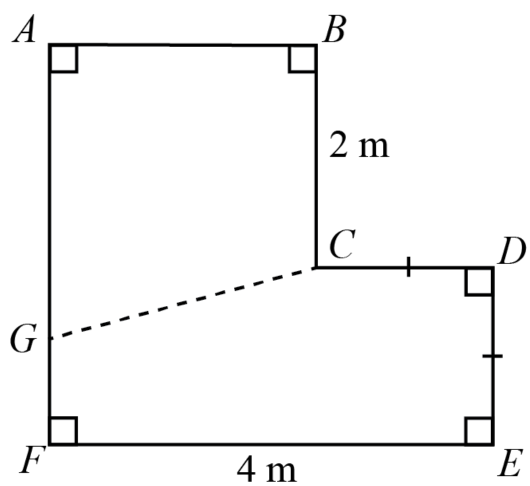


Where should G be located on AF in order to split the room into two smaller rooms of equal area?

Solution

Let x represent the length of CD, in metres. Since $DE = CD$, then $DE = x$.

Extend CD to intersect AF at H. This creates two rectangles $ABCH$ and $DEFH$ with $AB \parallel DH \parallel EF$. Also, $AB = EF - CD = 4 - x$.

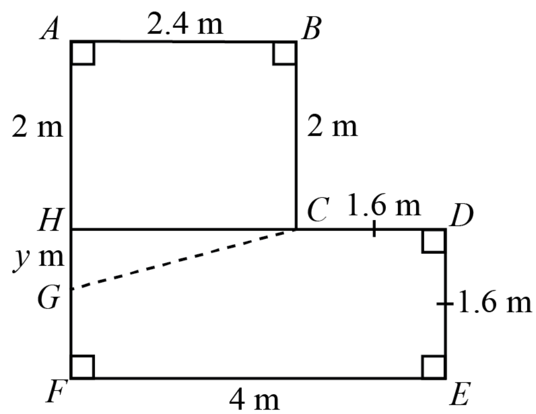
We can now find the value of x using areas.

$$\begin{aligned}
\text{Area } ABCDEF &= \text{Area } ABCH + \text{Area } DEFH \\
11.2 &= (AB \times BC) + (DE \times EF) \\
11.2 &= (4 - x)(2) + x(4) \\
11.2 &= 8 - 2x + 4x \\
3.2 &= 2x \\
1.6 &= x
\end{aligned}$$

Since $x = 1.6$ m, $CD = DE = 1.6$ m, and $AB = 4 - x = 2.4$ m. Also, $AH = BC = 2$ m, and $AF = DE + BC = 1.6 + 2 = 3.6$ m.

Now, the area of $ABCH$ is $2.4 \times 2 = 4.8$ m^2 and the area of area $DEFH$ is $4 \times 1.6 = 6.4$ m^2. Since $6.4 > 4.8$, then G must lie on HF.

Let y represent the length of GH, in metres. A diagram with updated information is below.



$ABCG$ is a trapezoid with opposite parallel sides $BC = 2$ and $AG = 2 + y$. AB is perpendicular to both BC and AG, and $AB = 2.4$ m. We also know that the area of trapezoid $ABCG$ is half the area of $ABCDEF$, so the area of trapezoid $ABCG$ is 5.6 m^2.

Therefore,

$$\begin{aligned}\text{Area of Trapezoid } ABCG &= \frac{AB \times (BC + AG)}{2}\\ 5.6 &= \frac{2.4 \times (2 + 2 + y)}{2}\\ 5.6 &= 1.2 \times (4 + y)\\ 5.6 &= 4.8 + 1.2y\\ 0.8 &= 1.2y\end{aligned}$$

Thus, $y = \frac{0.8}{1.2} = \frac{8}{12} = \frac{2}{3}$. Since $AG = 2 + y$, we have $AG = 2 + \frac{2}{3} = \frac{8}{3}$ m.

Also, since $GF = AF - AG$, we have $GF = 3.6 - \frac{8}{3} = \frac{18}{5} - \frac{8}{3} = \frac{54 - 40}{15} = \frac{14}{15}$ m.

Therefore, G should be positioned $\frac{14}{15}$ m from F, and $\frac{8}{3}$ m from A.

Problem of the Week

Problem D

Summing up a Sequence 1

The first term in a sequence is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

Elias writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. What is the smallest possible value of n?

24, 12, 6, ...

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Problem of the Week
Problem D and Solution
Summing up a Sequence 1

Problem

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Elias writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. What is the smallest possible value of n?

Solution

We will begin by finding more terms in the sequence. The first 14 terms of the sequence are 24, 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1.

If we continue the sequence, we will see that the terms 4, 2, and 1 will continue to repeat. Now we want to find the smallest possible value of n so that the sum of the terms in the sequence from term 1 to term n is at least 1000.

The sum of the first 8 terms is $24 + 12 + 6 + 3 + 10 + 5 + 16 + 8 = 84$. The sum of the repeating numbers is $4 + 2 + 1 = 7$. We want to determine the number of groups of repeating numbers. Let this be g. Suppose $84 + 7g = 1000$. Solving this gives $7g = 916$, so $g \approx 130.857$.

If $g = 130$, then the sum of the terms in the sequence is $84 + 7 \times 130 = 994$. This sequence contains the first 8 terms, plus 130 groups of the three repeating numbers. Therefore there are a total of $8 + 3 \times 130 = 398$ terms.

The 399^{th} term in the sequence will be 4, so the sum of the first 399 terms will be $994 + 4 = 998$.

The 400^{th} term in the sequence will be 2, so the sum of the first 400 terms will be $998 + 2 = 1000$. This is the smallest possible four-digit number, so the smallest possible value of n is 400.

Extension:

In 1937, the mathematician Lothat Collatz wondered if any sequence whose terms after the first are determined in this way would always eventually reach the number 1, regardless of which number you started with. This problem is actually still unsolved today and is called the Collatz Conjecture.

Problem of the Week

Problem D

Student to Student

At the beginning of the school year, the ratio of the number of Grade 10 students to the number of Grade 9 students at CEMC H.S. was $15 : 16$. By the end of the year, there were 30 more Grade 10 students and there were 20 fewer Grade 9 students, and the ratio of the number of Grade 10 students to the number of Grade 9 students was now $11 : 10$.

How many Grade 9 students and how many Grade 10 students were there at the beginning of the school year?

Problem of the Week
Problem D and Solution
Student to Student

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How many Grade 9 students and how many Grade 10 students were there at the beginning of the school year?

Solution

Originally, the ratio of the number of Grade 10 students to the number of Grade 9 students was 15 : 16. Therefore, we can let the number of Grade 10 students at the beginning of the year be $15n$ and the number of Grade 9 students at the beginning of the year be $16n$, for some integer n.

Thus, at the end of the year, there were $15n + 30$ Grade 10 students and $16n - 20$ Grade 9 students.

Since the ratio of the number of Grade 10 students at the end of the year to the number of Grade 9 students at the end of the year is 11 : 10, we have

$$\begin{aligned} \frac{15n+30}{16n-20} &= \frac{11}{10} \\ 150n + 300 &= 176n - 220 \\ 520 &= 26n \\ n &= 20 \end{aligned}$$

Therefore, there were $16n = 16(20) = 320$ Grade 9 students and $15n = 15(20) = 300$ Grade 10 students at the beginning of the school year.

Problem of the Week

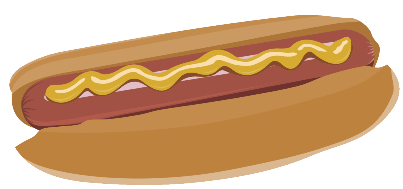
Problem D

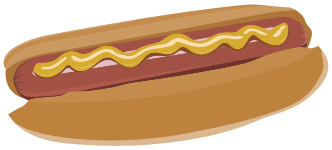
Lunchtime

Herman provides ketchup, relish, and mustard for the customers at his hot dog stand. During the lunch rush one day, he recorded how many customers had each of these three condiments. He observed the following:

- The total number of customers was 125, and each ordered a single hot dog.
- 82 customers had ketchup, 47 had relish, and 80 had mustard.
- 32 customers had mustard and ketchup, but not relish.
- 5 customers had mustard and relish, but not ketchup.
- The number of customers who had all three of the condiments was double the number of customers who had none of the condiments.
- The number of customers who had exactly two of the condiments was double the number of customers who had all three of the condiments.

How many customers had ketchup, relish, and mustard on their hot dog?





Problem of the Week
Problem D and Solution
Lunchtime

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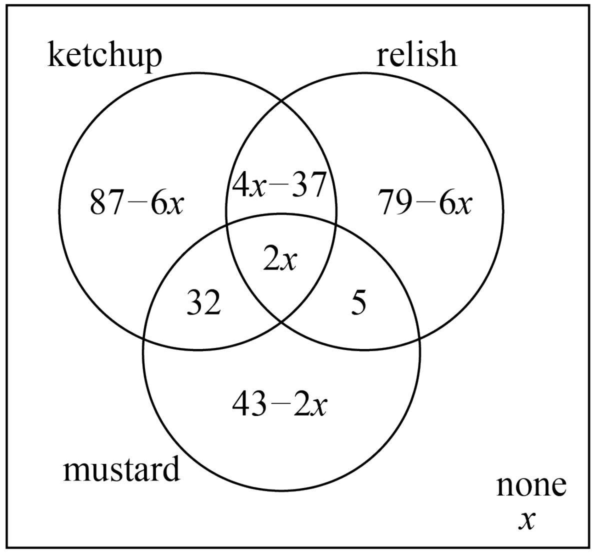
How many customers had ketchup, relish, and mustard on their hot dog?

Solution

Let x be the number of customers who had none of the condiments. Then $2x$ customers had all three of the condiments, and $4x$ customers had exactly two of the condiments.

From the given information, since 32 customers had mustard and ketchup but not relish, and 5 customers had mustard and relish but not ketchup, it follows that $4x - 32 - 5 = 4x - 37$ customers had ketchup and relish, but not mustard.

Also, since 82 customers in total had ketchup, then $82 - 32 - 2x - (4x - 37) = 87 - 6x$ customers had only ketchup. Similarly, since 47 customers in total had relish, then $47 - 5 - 2x - (4x - 37) = 79 - 6x$ customers had only relish. As well, since 80 customers in total had mustard, then $80 - 32 - 2x - 5 = 43 - 2x$ customers had only mustard. We summarize this information in the following Venn diagram.



Since there were 125 customers in total,

$$\begin{aligned}125 &= (87 - 6x) + (4x - 37) + 2x + 32 + (79 - 6x) + 5 + (43 - 2x) + x \\ 125 &= 209 - 7x \\ 7x &= 84 \\ x &= 12\end{aligned}$$

Therefore, $2x = 24$ customers had ketchup, relish, and mustard on their hot dog.

Problem of the Week

Problem D

Sticker Situation

Kendi has a large collection of vinyl stickers, and each sticker has an animal on it or an emoji on it (but not both). In this collection, 300 of the stickers have animals on them, and 1 out of 5 of all of the stickers have animals on them. She would like to add more stickers to her collection so that there are 3 stickers with animals on them out of every 10 stickers.

If she can buy the stickers in packages of 60 stickers where 21 are animal stickers and the remaining are emoji stickers, how many whole packages does she need to buy?

Problem of the Week
Problem D and Solution
Sticker Situation

Problem

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If she can buy the stickers in packages of 60 stickers where 21 are animal stickers and the remaining are emoji stickers, how many whole packages does she need to buy?

Solution

There were initially 300 animal stickers, and there was 1 animal sticker for every 5 stickers. This means that 4 out of 5 stickers were emoji stickers. Therefore, there were four times as many emoji stickers as animal stickers. That is, there were $4 \times 300 = 1200$ emoji stickers and a total of $300 + 1200 = 1500$ stickers.

Each package contains 21 animal stickers and 39 emoji stickers, for a total of $21 + 39 = 60$ stickers.

Let n represent the number of additional whole packages required to add to this collection so that there are 3 animal stickers out of every 10 of the stickers. By purchasing n packages, she is adding $60n$ stickers to her collection, of which $21n$ are animal stickers. Thus, she will have a total of $1500 + 60n$ stickers, of which $300 + 21n$ are animal stickers.

If 3 out of 10 of the stickers in her collection are animal stickers, then we have

$$\begin{aligned}
\frac{\text{the number of animal stickers}}{\text{the total number of stickers}} &= \frac{3}{10} \\
\frac{300 + 21n}{1500 + 60n} &= \frac{3}{10} \\
10(300 + 21n) &= 3(1500 + 60n) \\
3000 + 210n &= 4500 + 180n \\
30n &= 1500 \\
n &= 50
\end{aligned}$$

Therefore, 50 additional packages of stickers must be purchased so that 3 out of 10 of the stickers in her collection are animal stickers.

We can check this. After purchasing 50 additional packages of stickers, there would be $300 + 21(50) = 1350$ animal stickers and a total of $1500 + 60(50) = 4500$ stickers. Then, the ratio of animal stickers to the total number of stickers is $\frac{1350}{4500} = \frac{3}{10}$, as required.

Problem of the Week

Problem D

Birthday Cake Trouble

Harri is the manager of a pet store. For his birthday, his employees decided to surprise him with a birthday cake. The cake cost \$65.40 and everyone agreed to split the cost evenly. However, when it came time to collect the money, three of the employees were nowhere to be found. This meant that everyone else had to pay an additional \$1.09 to cover the cost of the cake. How many employees does Harri have at the pet store?

Problem of the Week

Problem D and Solution

Birthday Cake Trouble

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Solution

Solution 1

Let n represent the number of employees that Harri has at the pet store. Then $(n-3)$ represents the number of employees who actually paid for the cake.

The n employees had each agreed to pay $\frac{\$65.40}{n}$. However, $(n-3)$ employees actually each paid $\frac{\$65.40}{n-3}$, which was more than the amount they had originally agreed to. The difference between the two amounts is \$1.09. It follows that

$$\begin{aligned}
\frac{65.40}{n-3} - \frac{65.40}{n} &= 1.09 \\
\frac{65.40}{n-3}(n)(n-3) - \frac{65.40}{n}(n)(n-3) &= 1.09(n)(n-3) \\
65.40n - 65.40(n-3) &= 1.09(n)(n-3) \\
65.40n - 65.40n + 196.2 &= 1.09(n)(n-3) \\
\frac{196.2}{1.09} &= n(n-3) \\
180 &= n(n-3)
\end{aligned}$$

From here we see that we are looking for two positive integers that differ by 3 and multiply to 180. We notice that $15 \times 12 = 180$, and $15 - 12 = 3$. In fact these are the only two positive integers that differ by 3 and multiply to 180. It follows that $n = 15$. Thus, Harri has 15 employees at the pet store.

Solution 2

This solution builds onto Solution 1 by solving the problem algebraically. Note that this level of mathematics is often not taught until grade 10.

Start with Solution 1 and proceed until you reach $180 = n(n-3)$. From there,

$$\begin{aligned}
180 &= n^2 - 3n \\
0 &= n^2 - 3n - 180 \\
0 &= (n-15)(n+12)
\end{aligned}$$

Therefore, $n = 15$ or $n = -12$. However, since $n > 0$, it follows that $n = 15$. Thus, Harri has 15 employees at the pet store.

Solution 3

Let n represent the number of employees that Harri has at the pet store. Then $(n-3)$ represents the number of employees who actually paid for the cake. The cost of the cake was \$65.40, or 6540 cents. Since 3 of the employees didn't pay, and at least one employee did pay, then we can assume there are at least 4 employees in total.

We check integer values of n, starting with $n = 4$, and determine the difference between the cost per person when there are n people compared to when there are $(n-3)$ people, until we find a difference of 109 cents. This is summarized in the table below.

Number of Employees (n)	Original Amount, in cents, per Employee $\left(\frac{6540}{n}\right)$	Number of Employees who Paid $(n-3)$	Amount Actually Paid, in cents, per Employee $\left(\frac{6540}{n-3}\right)$	Difference in Amounts, in cents $\left(\frac{6540}{n-3} - \frac{6540}{n}\right)$
4	$\frac{6540}{4} = 1635$	$4-3=1$	$\frac{6540}{1} = 6540$	$6540 - 1635 = 4905$
5	$\frac{6540}{5} = 1308$	$5-3=2$	$\frac{6540}{2} = 3270$	$3270 - 1308 = 1962$
6	$\frac{6540}{6} = 1090$	$6-3=3$	$\frac{6540}{3} = 2180$	$2180 - 1090 = 1090$
7	$\frac{6540}{7} \approx 934.29$	$7-3=4$	$\frac{6540}{4} = 1635$	$1635 - 934.29 = 700.71$
8	$\frac{6540}{8} = 817.5$	$8-3=5$	$\frac{6540}{5} = 1308$	$1308 - 817.5 = 490.5$
9	$\frac{6540}{9} \approx 726.67$	$9-3=6$	$\frac{6540}{6} = 1090$	$1090 - 726.67 = 363.33$
10	$\frac{6540}{10} = 654$	$10-3=7$	$\frac{6540}{7} \approx 934.29$	$934.29 - 654 = 280.29$
11	$\frac{6540}{11} \approx 594.55$	$11-3=8$	$\frac{6540}{8} = 817.5$	$817.5 - 594.55 = 222.95$
12	$\frac{6540}{12} = 545$	$12-3=9$	$\frac{6540}{9} \approx 726.67$	$726.67 - 545 = 181.67$
13	$\frac{6540}{13} \approx 503.08$	$13-3=10$	$\frac{6540}{10} = 654$	$654 - 503.08 = 150.92$
14	$\frac{6540}{14} \approx 467.14$	$14-3=11$	$\frac{6540}{11} \approx 594.55$	$594.55 - 467.14 = 127.14$
15	$\frac{6540}{15} = 436$	$15-3=12$	$\frac{6540}{12} = 545$	$545 - 436 = 109$

Therefore, when $n = 15$, the difference in amounts is 109 cents, or \$1.09, as desired. Since the difference is decreasing as n is increasing, this is the only possible value of n. Thus, Harri has 15 employees at the pet store.

Problem of the Week

Problem D

1225 is SUMthing Special

Did you know that 1225 can be written as the sum of ten consecutive integers?

That is,

$$1225 = 118 + 119 + 120 + 121 + 122 + 123 + 124 + 125 + 126 + 127$$

The notation below illustrates a mathematical short form used for writing the above sum. This notation is called *Sigma Notation*.

$$\sum_{i=118}^{127} i = 1225$$

How many ways can the number 1225 be expressed as the sum of an **odd** number of consecutive positive integers?

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Problem of the Week
Problem D and Solution
1225 is SUMthing Special

Problem
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$$1225 = 118 + 119 + 120 + 121 + 122 + 123 + 124 + 125 + 126 + 127$$

How many ways can the number 1225 be expressed as the sum of an **odd** number of consecutive positive integers?

Solution

We will use the following idea to solve this problem: If there exists an odd number, k, of consecutive integers that sum to 1225, then k is a divisor of 1225.

Furthermore, if $kn = 1225$, then n is the mean (average) of the k integers, and will appear in the middle of the sequence of numbers summing to 1225.

Why is this true? Let's first consider $k = 5$.

Five consecutive integers can be expressed as $n - 2$, $n - 1$, n, $n + 1$, and $n + 2$, where n is an integer.
Their sum is $(n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 5n$.
Therefore, $5n = 1225$ and $n = 245$. Thus, the middle term in the sum is 245, and the series is $243 + 244 + 245 + 246 + 247 = 1225$.
In general, if there are k consecutive integers, where k is odd, and n is the middle number in the sum, then there are $\frac{k-1}{2}$ integers less than n in the sum and $\frac{k-1}{2}$ integers greater than n in the sum. Furthermore, the first integer in the sum is $n - \frac{k-1}{2}$, the last integer in the sum is $n + \frac{k-1}{2}$, and we can write the sum of these integers in this way:

$$\left(n - \frac{k-1}{2}\right) + \cdots + (n-3) + (n-2) + (n-1) + n + (n+1) + (n+2) + (n+3) + \cdots + \left(n + \frac{k-1}{2}\right)$$

This simplifies to kn. Thus, if this sum is equal to 1225, then $kn = 1225$ and so k is an odd divisor of 1225.

Since $1225 = 5^2 7^2$, the positive divisors of 1225 are 1, 5, 7, 25, 35, 49, 175, 245 and 1225, which are all odd.

For each odd divisor, k, of 1225, we determine $n = \frac{1225}{k}$, which will be the middle term in the sum. The k integers that sum to 1225 will then be $\left(n - \frac{k-1}{2}\right) + \cdots + n + \cdots + \left(n + \frac{k-1}{2}\right)$. This is summarized in the table below.

Number of Integers (k)	Middle Integer (n)	Sum of Integers
1	1225	1225
5	245	$243 + 244 + 245 + 246 + 247$
7	175	$172 + 173 + 174 + 175 + 176 + 177 + 178$
25	49	$37 + 38 + \cdots + 49 + \cdots + 60 + 61$
35	35	$18 + 19 + \cdots + 35 + \cdots + 51 + 52$
49	25	$1 + 2 + \cdots + 25 + \cdots + 48 + 49$
175	7	$(-80) + (-70) + \cdots + 7 + \cdots + 93 + 94$
245	5	$(-117) + (-116) + \cdots + 5 + \cdots + 126 + 127$
1225	1	$(-611) + (-610) + \cdots + 1 + \cdots + 612 + 613$

Note that all integers in the sum are positive for $k = 1, 5, 7, 25, 35, 49$. For $k = 175, 245, 1225$, there are negative integers in the sum.

Thus, there are six ways to express 1225 as the sum of an odd number of consecutive positive integers.

Extension: Determine the number of ways the number 1225 can be expressed as the sum of an **even** number of consecutive positive integers.

Problem of the Week

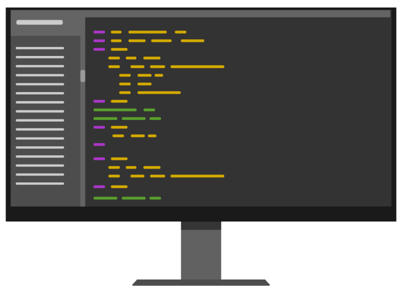
Problem D

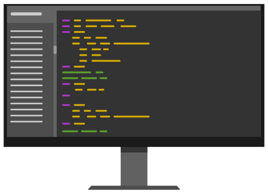
Another Program

To practice her programming skills, Tessa wrote a program that takes an input number, squares it, adds a constant value, multiplies the result by another constant value, and then outputs the result.

When the input is 8, the output is 204. When the input is 3, the output is 39.

What is the output when the input is 5?





Problem of the Week
Problem D and Solution
Another Program

Problem

To practice her programming skills, Tessa wrote a program that takes an input number, squares it, adds a constant value, multiplies the result by another constant value, and then outputs the result.

When the input is 8, the output is 204. When the input is 3, the output is 39.

What is the output when the input is 5?

Solution

Let the first constant value be k and the second constant value be p. Then Tessa's program takes an input number, squares it, adds k, multiplies the result by p, and then outputs the result.

Since an input of 8 gives an output of 204,

$$\begin{aligned}(8^2 + k) \times p &= 204 \\ p &= \frac{204}{64 + k} \qquad (1)\end{aligned}$$

Similarly, since an input of 3 gives an output of 39,

$$\begin{aligned}(3^2 + k) \times p &= 39 \\ p &= \frac{39}{9 + k} \qquad (2)\end{aligned}$$

From equations (1) and (2), we can conclude the following.

$$\begin{aligned}\frac{204}{64 + k} &= \frac{39}{9 + k} \\ 204(9 + k) &= 39(64 + k) \\ 1836 + 204k &= 2496 + 39k \\ 165k &= 660 \\ k &= \frac{660}{165} = 4\end{aligned}$$

Then $p = \dfrac{39}{9 + k} = \dfrac{39}{9 + 4} = 3$.

Now that we have determined the values of k and p, we can determine the output when the input is 5.

$$(5^2 + 4) \times 3 = 29 \times 3 = 87$$

Thus, the output is 87 when the input is 5.

Computational Thinking (C)

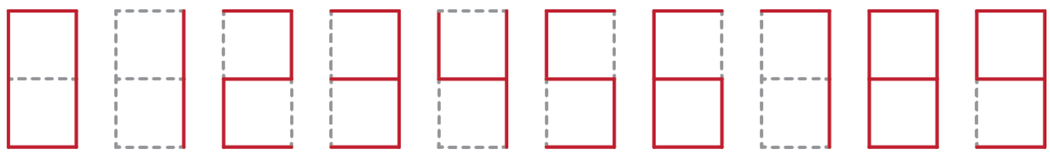
Take me to the cover

Problem of the Week

Problem D

The Clock Works

Halina's clock uses a digital LED display where each digit is represented by seven LED segments that are either on or off, as shown.



Sometimes some of the LED segments stop working. When the top-most horizontal LED segment stopped working, both the digit 1 and the digit 7 appeared as shown. This was a problem because Halina couldn't distinguish between them.

Halina replaced the broken LED segment, but then a week later found that a different LED segment had stopped working. However, this time, she was still able to distinguish between all ten digits.

What is the largest number of LED segments that can be broken at the same time, while still allowing Halina to distinguish between all ten digits?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Problem of the Week
Problem D and Solution
The Clock Works

Problem

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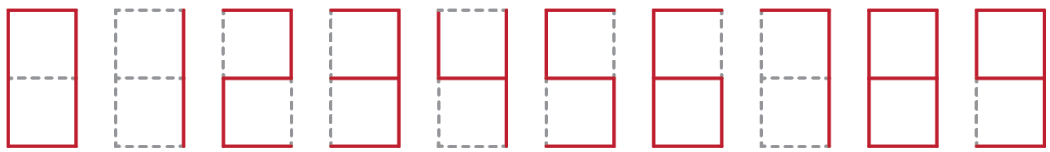

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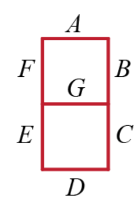
What is the largest number of LED segments that can be broken at the same time, while still allowing Halina to distinguish between all ten digits?

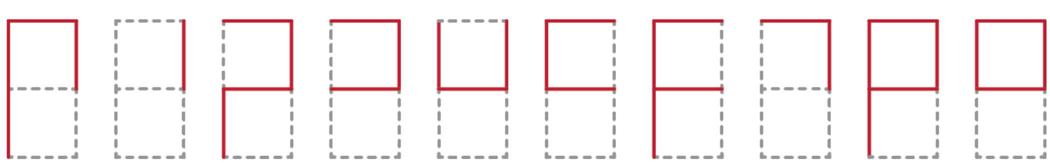
Solution

First we will label the seven LED segments as shown.



If segment A is broken, Halina cannot distinguish between digits 1 and 7.
If segment B is broken, Halina cannot distinguish between digits 6 and 8.
If segment E is broken, Halina cannot distinguish between digits 5 and 6, and also between digits 8 and 9.
If segment F is broken, Halina cannot distinguish between digits 3 and 9.
If segment G is broken, Halina cannot distinguish between digits 0 and 8.

If both segments C and D are broken, the ten digits would appear as shown.



Since each of these digits are unique, Halina can distinguish between them. Therefore, at most 2 LED segments can be broken at the same time.

Problem of the Week

Problem D

Four on the ION

On Tuesday afternoon, Fred, Gerlach, Hamza, and Iuliana got on the same ION light rail train at University of Waterloo station after their band practice. Each person was carrying a different instrument and each person got off at a different station. Using the clues below, determine what instrument each person was carrying, and in which order they got off the train.

(1) Fred got off the train after Gerlach.

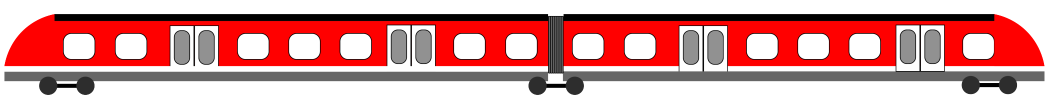
(2) The person carrying the saxophone was *not* the first to get off the train.

(3) The person who plays the guitar is *not* Iuliana.

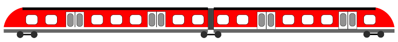
(4) The person with the tuba was the last to get off the train.

(5) Fred is *not* the person who plays the tuba.

(6) Iuliana got off the train before the person with the clarinet.



Problem of the Week



Problem D and Solution

Four on the ION

Problem

On Tuesday afternoon, Fred, Gerlach, Hamza, and Iuliana got on the same ION light rail train at University of Waterloo station after their band practice. Each person was carrying a different instrument and each person got off at a different station. Using the clues below, determine what instrument each person was carrying, and in which order they got off the train.

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(4) The person with the tuba was the last to get off the train.

(5) Fred is *not* the person who plays the tuba.

(6) Iuliana got off the train before the person with the clarinet.

Solution

We can use a table to summarize the information in the clues. We will start by filling in the information from clues (2), (4), and (5). If we know that something does not go in a particular cell, we will also include that in the table.

	1^{st} to get off	2^{nd} to get off	3^{rd} to get off	4^{th} to get off
Name				*not* Fred
Instrument	*not* saxophone			tuba

Next we look at clues (3) and (6). From clue (6) we can conclude that the person with the clarinet was not the 1^{st} person to get off the train. Since we know that the 1^{st} person to get off the train was also not carrying the saxophone, and we have already placed the tuba, we can conclude that the 1^{st} person to get off the train was carrying a guitar.

From clue (3), we know that Iuliana was not the 1^{st} person to get off the train. From clue (6) we know that Iuliana got off the train before the person with the clarinet. It follows that Iuliana got off the train 2^{nd}, and the person with the clarinet got off the train 3^{rd}.

	1^{st} to get off	2^{nd} to get off	3^{rd} to get off	4^{th} to get off
Name		Iuliana		*not* Fred
Instrument	guitar		clarinet	tuba

Finally, since the only instrument not yet placed is the saxophone, we can conclude that Iuliana was carrying a saxophone. From clue (1), it follows that Gerlach got off the train 1^{st}, Fred got off the train 3^{rd}, and Hamza got off the train 4^{th}. The final solution is shown.

	1^{st} to get off	2^{nd} to get off	3^{rd} to get off	4^{th} to get off
Name	Gerlach	Iuliana	Fred	Hamza
Instrument	guitar	saxophone	clarinet	tuba

Problem of the Week

Problem D

Again in Reverse

A *palindrome* is a word or phrase which reads the same forwards and backwards, ignoring spaces and punctuation. Numbers which remain the same when the digits are reversed are also considered to be palindromes. For example 9 357 539, 6116, and 2 are all palindromes.

How many positive integers less than 1 000 000 are palindromes?

No lemons, no melon

Was it a car or a cat I saw?

Problem of the Week

Problem D and Solution

Again in Reverse

Problem

A *palindrome* is a word or phrase which reads the same forwards and backwards, ignoring spaces and punctuation. Numbers which remain the same when the digits are reversed are also considered to be palindromes. For example 9 357 539, 6116, and 2 are all palindromes.

How many positive integers less than 1 000 000 are palindromes?

Solution

We consider cases, based on the number of digits in the palindrome. Since we are looking for positive integers less than 1 000 000, the minimum number of digits is one and the maximum is six.

- **Case 1:** One-digit palindromes
  Each of the integers from 1 to 9 is a palindrome. Thus, there are 9 one-digit palindromes.

- **Case 2:** Two-digit palindromes
  In order to be a palindrome, the two digits must be the same. Therefore, the integer must be of the form aa, where a is an integer from 1 to 9. Thus, there are 9 two-digit palindromes.

- **Case 3:** Three-digit palindromes
  In order to be a palindrome, the first and last digits must be the same. Therefore, the integer must be of the form aba, where a is an integer from 1 to 9 and b is an integer from 0 to 9. There are 9 choices for a, and for each of these choices there are 10 choices for b. Thus, there are $9 \times 10 = 90$ three-digit palindromes.

- **Case 4:** Four-digit palindromes
  In order to be a palindrome, the first and last digits must be the same, and the second and third digits must be the same. Therefore, the integer must be of the form $abba$, where a is an integer from 1 to 9 and b is an integer from 0 to 9. As with the previous case, there are 9 choices for a, and for each of these choices there are 10 choices for b. Thus, there are $9 \times 10 = 90$ four-digit palindromes.

- **Case 4:** Five-digit palindromes
  In order to be a palindrome, the first and last digits must be the same, and the second and fourth digits must be the same. Therefore, the integer must be of the form $abcba$, where a is an integer from 1 to 9 and b and c are integers from 0 to 9. There are 9 choices for a, for each of these choices there are 10 choices for b, and for each of these choices there are 10 choices for c. Thus, there are $9 \times 10 \times 10 = 900$ five-digit palindromes.

- **Case 4:** Six-digit palindromes
  In order to be a palindrome, the first and last digits must be the same, the second and fifth digits must be the same, and the third and fourth digits must be the same. Therefore, the integer must be of the form $abccba$, where a is an integer from 1 to 9 and b and c are integers from 0 to 9. As with the previous case, there are 9 choices for a, for each of these choices there are 10 choices for b, and for each of these choices there are 10 choices for c. Thus, there are $9 \times 10 \times 10 = 900$ six-digit palindromes.

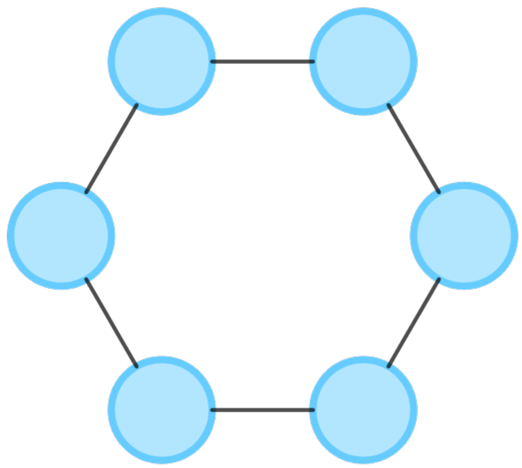
Thus, there are $9 + 9 + 90 + 90 + 900 + 900 = 1998$ palindromes less than $1\,000\,000$.

Problem of the Week

Problem D

Might I Win Every Time?

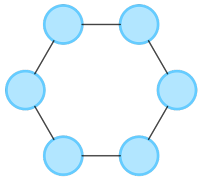
In this two-player game, the vertices of a regular hexagon are each covered by a circle.



On a turn, a player may either initial one circle or initial two adjacent circles. (The two adjacent circles must be directly connected by an outside edge of the hexagon.) Players alternate turns. The player initialing the last circle or last pair of adjacent circles is the winner.

Two players, Cameron and Dale, play the game with Cameron going first.

One of the players, Cameron or Dale, can always win the game no matter what moves the other player makes. Describe which player can always win and the winning strategy for that player.



Problem of the Week
Problem D and Solution
Might I Win Every Time?

Problem

In this two-player game, the vertices of a regular hexagon are each covered by a circle.

On a turn, a player may either initial one circle or initial two adjacent circles. (The two adjacent circles must be directly connected by an outside edge of the hexagon.) Players alternate turns. The player initialing the last circle or last pair of adjacent circles is the winner.

Two players, Cameron and Dale, play the game with Cameron going first.

One of the players, Cameron or Dale, can always win the game no matter what moves the other player makes. Describe which player can always win and the winning strategy for that player.

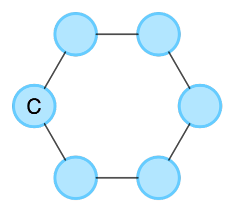
Solution

We will show that Dale can always win the game, and will describe Dale's winning strategy.

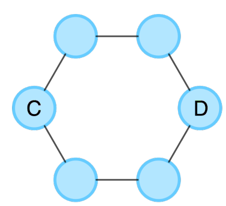
There are two types of possible first moves for Cameron: Cameron could initial exactly one blank circle, or Cameron could initial two adjacent blank circles.

- **Case 1:** Cameron initials exactly one blank circle on his first move.

  We can rotate the hexagon without changing the game, so we will assume that Cameron initials the circle on the left with a 'C', as shown.

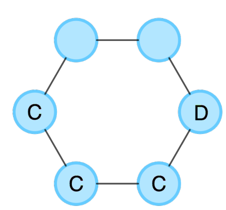
  

  In this case, Dale should initial the circle on the opposite side of the hexagon with a 'D'.

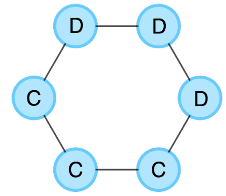
  

  There are now two types of possible second moves for Cameron: Cameron could initial exactly one blank circle, or Cameron could initial two adjacent blank circles.

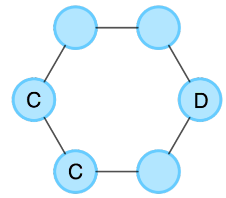
  Suppose Cameron initials two adjacent circles, either at the top or bottom, on his second turn. The situation where he initials the bottom two circles is shown.

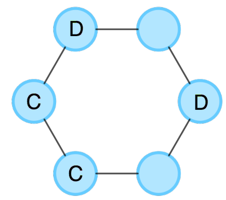
Dale should then initial the two remaining adjacent circles on the opposite side to win the game.



Instead, suppose Cameron initials only one circle on his second turn. The situation where he initials the bottom left circle is shown.



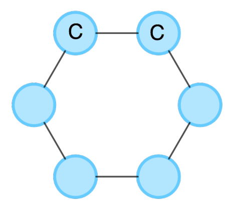
Dale should then initial one of the two adjacent unmarked circles on the other side of the hexagon. The situation where he initials the top left circle is shown.



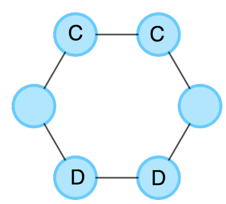
Now there are two non-adjacent unmarked circles left. On his third turn, Cameron must initial one of these unmarked circles. Dale then wins the game by initialing the last unmarked circle.

- **Case 2:** Cameron initials two adjacent blank circles on his first move.

Again, we can rotate the hexagon without changing the game, so we will assume that Cameron initials the two unmarked circles at the top.



In this case, Dale should initial the two adjacent unmarked circles at the bottom.



Now there are two non-adjacent unmarked circles left, and Cameron must initial exactly one of these unmarked circles. Dale then initials the last unmarked circle to win the game.

We have considered all of the possible cases and have shown that Dale has a winning strategy in each case. Dale should "copy" Cameron by initialing the same number of unmarked circles as Cameron, but on the opposite side of the hexagon. If Dale follows this strategy, Dale will always win.

For Further Thought:

Without changing the rules of the game, how would the strategy change if instead of a hexagon with 6 circles, there were a heptagon with 7 circles?

Data Management (D)

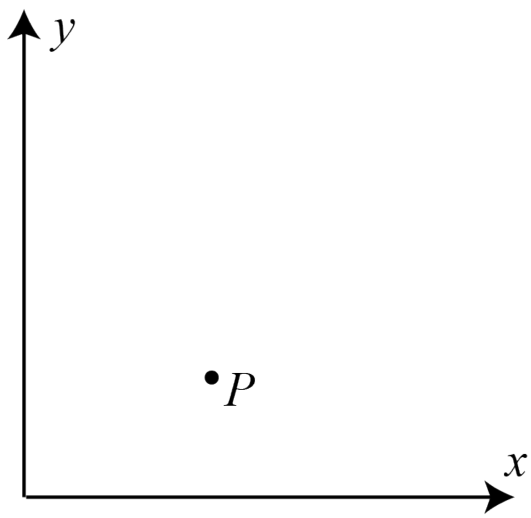
Take me to the cover

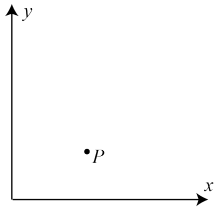
Problem of the Week

Problem D

Slope on a Plane

Percy drew x and y axes on grid paper and then plotted the point $P(8, 5)$. Quinlan then chose a different point, Q, and said its coordinates were each positive integers less than or equal to 20. Determine the probability that the slope of PQ is 0, 1, or 2.





Problem of the Week

Problem D and Solution

Slope on a Plane

Problem

Percy drew x and y axes on grid paper and then plotted the point $P(8, 5)$. Quinlan then chose a different point, Q, and said its coordinates were each positive integers less than or equal to 20. Determine the probability that the slope of PQ is 0, 1, or 2.

Solution

First, suppose the slope of PQ is 0. Then Q must have a y-coordinate of 5. Since the coordinates of Q are each positive integers less than or equal to 20, the smallest possible x-coordinate is 1 and the largest is 20. Thus, there are $20 - 1 + 1 = 20$ possible points. However, this includes the point P, so there are $20 - 1 = 19$ possibilities for Q such that PQ has slope 0.

Next, suppose the slope of PQ is 1. Let the coordinates of Q be (a, b). Then

$$\begin{aligned} \frac{b-5}{a-8} &= 1 \\ b - 5 &= a - 8 \\ b &= a - 3 \end{aligned}$$

Since a and b are each positive integers less than or equal to 20, the point with the smallest possible value for a is $(4, 1)$. Similarly, the point with the largest possible value for a is $(20, 17)$. Since $b = a - 3$, there is a possible value for b for each value of a between 4 and 20. Thus, there are $20 - 4 + 1 = 17$ possible points. However, this includes the point P, so there are $17 - 1 = 16$ possibilities for Q such that PQ has slope 1.

Finally, suppose the slope of PQ is 2. Let the coordinates of Q be (a, b). Then

$$\begin{aligned} \frac{b-5}{a-8} &= 2 \\ b - 5 &= 2(a - 8) \\ b - 5 &= 2a - 16 \\ b &= 2a - 11 \end{aligned}$$

Since a and b are each positive integers less than or equal to 20, the point with the smallest possible value for a is $(6, 1)$. To determine the point with the largest possible value for a, we first notice that if $a = 20$, then $b > 20$. Then we can set $b = 2a - 11 < 20$. Thus $2a < 31$, or $a < 15.5$. It follows that the point with the largest possible value for a is $(15, 19)$. Since $b = 2a - 11$, there is a possible value for b for each value of a between 6 and 15. Thus, there are $15 - 6 + 1 = 10$ possible points. However, this includes the point P, so there are $10 - 1 = 9$ possibilities for Q such that PQ has slope 2.

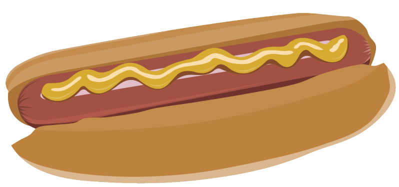
Thus, the total number of possibilities for Q such that PQ has slope 0, 1, or 2 is $19 + 16 + 9 = 44$. The total number of possibilities for Q is $20 \times 20 - 1 = 399$. Thus, the probability that the slope of PQ is 0, 1, or 2 is equal to $\frac{44}{399}$, or approximately 11%.

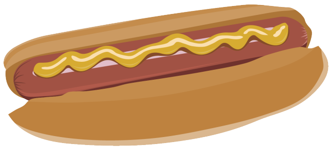
Problem of the Week
Problem D
Lunchtime

Herman provides ketchup, relish, and mustard for the customers at his hot dog stand. During the lunch rush one day, he recorded how many customers had each of these three condiments. He observed the following:

- The total number of customers was 125, and each ordered a single hot dog.
- 82 customers had ketchup, 47 had relish, and 80 had mustard.
- 32 customers had mustard and ketchup, but not relish.
- 5 customers had mustard and relish, but not ketchup.
- The number of customers who had all three of the condiments was double the number of customers who had none of the condiments.
- The number of customers who had exactly two of the condiments was double the number of customers who had all three of the condiments.

How many customers had ketchup, relish, and mustard on their hot dog?





Problem of the Week
Problem D and Solution
Lunchtime

Problem

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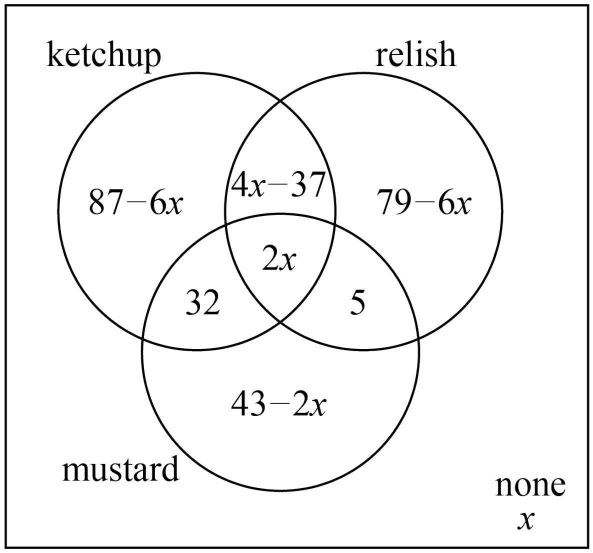
How many customers had ketchup, relish, and mustard on their hot dog?

Solution

Let x be the number of customers who had none of the condiments. Then $2x$ customers had all three of the condiments, and $4x$ customers had exactly two of the condiments.

From the given information, since 32 customers had mustard and ketchup but not relish, and 5 customers had mustard and relish but not ketchup, it follows that $4x - 32 - 5 = 4x - 37$ customers had ketchup and relish, but not mustard.

Also, since 82 customers in total had ketchup, then $82 - 32 - 2x - (4x - 37) = 87 - 6x$ customers had only ketchup. Similarly, since 47 customers in total had relish, then $47 - 5 - 2x - (4x - 37) = 79 - 6x$ customers had only relish. As well, since 80 customers in total had mustard, then $80 - 32 - 2x - 5 = 43 - 2x$ customers had only mustard. We summarize this information in the following Venn diagram.



Since there were 125 customers in total,

$$\begin{aligned}125 &= (87-6x)+(4x-37)+2x+32+(79-6x)+5+(43-2x)+x\\125 &= 209-7x\\7x &= 84\\x &= 12\end{aligned}$$

Therefore, $2x = 24$ customers had ketchup, relish, and mustard on their hot dog.

Geometry & Measurement (G)

Take me to the cover

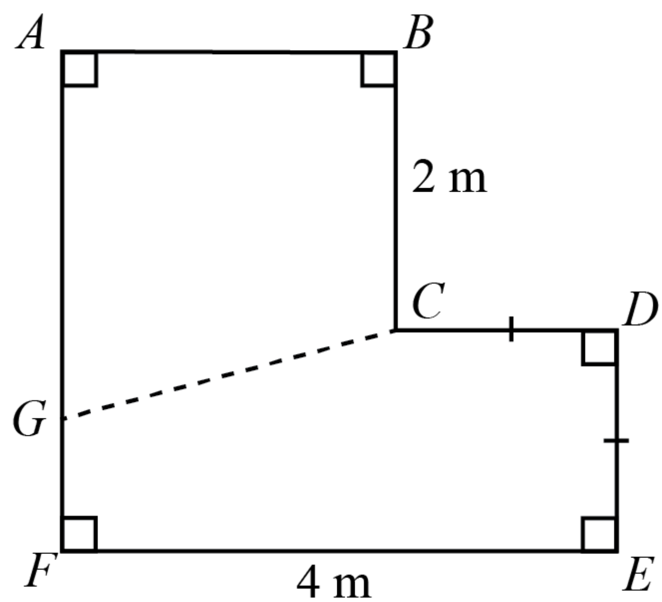
Problem of the Week

Problem D

Dividing Line

The Bobsie twins share an L-shaped room. The area of the entire room is $11.2\ \text{m}^2$. The twins are not getting along, so their parents decide to partition the room with tape so that each child has exactly the same area.

The layout of their room is represented by $ABCDEF$ in the diagram. The partitioning tape, indicated by a dashed line, will travel from C to a point G on AF.



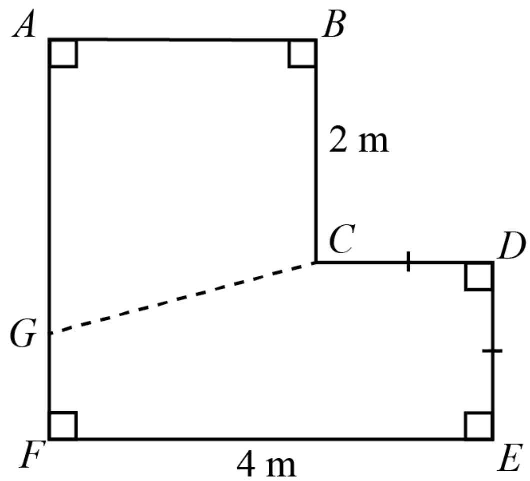
Where should G be located on AF in order to split the room into two smaller rooms of equal area?

Problem of the Week
Problem D and Solution
Dividing Line

Problem

The Bobsie twins share an L-shaped room. The area of the entire room is 11.2 m^2. The twins are not getting along, so their parents decide to partition the room with tape so that each child has exactly the same area.

The layout of their room is represented by $ABCDEF$ in the diagram. The partitioning tape, indicated by a dashed line, will travel from C to a point G on AF.



Where should G be located on AF in order to split the room into two smaller rooms of equal area?

Solution

Let x represent the length of CD, in metres. Since $DE = CD$, then $DE = x$.

Extend CD to intersect AF at H. This creates two rectangles $ABCH$ and $DEFH$ with $AB \parallel DH \parallel EF$. Also, $AB = EF - CD = 4 - x$.

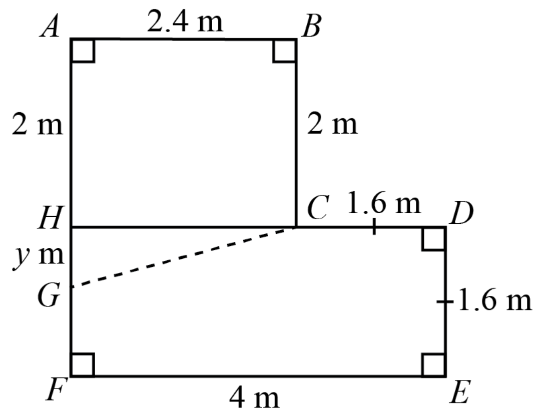
We can now find the value of x using areas.

$$\begin{aligned}\text{Area } ABCDEF &= \text{Area } ABCH + \text{Area } DEFH \\ 11.2 &= (AB \times BC) + (DE \times EF) \\ 11.2 &= (4-x)(2) + x(4) \\ 11.2 &= 8 - 2x + 4x \\ 3.2 &= 2x \\ 1.6 &= x\end{aligned}$$

Since $x = 1.6$ m, $CD = DE = 1.6$ m, and $AB = 4 - x = 2.4$ m. Also, $AH = BC = 2$ m, and $AF = DE + BC = 1.6 + 2 = 3.6$ m.

Now, the area of $ABCH$ is $2.4 \times 2 = 4.8$ m^2 and the area of area $DEFH$ is $4 \times 1.6 = 6.4$ m^2. Since $6.4 > 4.8$, then G must lie on HF.

Let y represent the length of GH, in metres. A diagram with updated information is below.



$ABCG$ is a trapezoid with opposite parallel sides $BC = 2$ and $AG = 2 + y$. AB is perpendicular to both BC and AG, and $AB = 2.4$ m. We also know that the area of trapezoid $ABCG$ is half the area of $ABCDEF$, so the area of trapezoid $ABCG$ is 5.6 m^2.

Therefore,

$$\begin{aligned} \text{Area of Trapezoid } ABCG &= \frac{AB \times (BC + AG)}{2} \\ 5.6 &= \frac{2.4 \times (2 + 2 + y)}{2} \\ 5.6 &= 1.2 \times (4 + y) \\ 5.6 &= 4.8 + 1.2y \\ 0.8 &= 1.2y \end{aligned}$$

Thus, $y = \frac{0.8}{1.2} = \frac{8}{12} = \frac{2}{3}$. Since $AG = 2 + y$, we have $AG = 2 + \frac{2}{3} = \frac{8}{3}$ m.

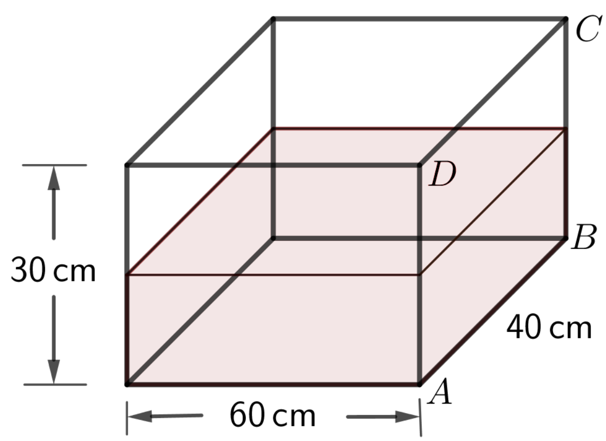
Also, since $GF = AF - AG$, we have $GF = 3.6 - \frac{8}{3} = \frac{18}{5} - \frac{8}{3} = \frac{54 - 40}{15} = \frac{14}{15}$ m.

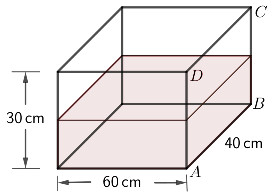
Therefore, G should be positioned $\frac{14}{15}$ m from F, and $\frac{8}{3}$ m from A.

Problem of the Week
Problem D
Tilt!

Troy has a container in the shape of a rectangular prism with a 40 cm by 60 cm base and a height of 30 cm. He labels the vertices of a 40 cm by 30 cm side face A, B, C, and D, with A and B being vertices of the base face too. He then puts some water in the container and tilts the container along AB until the water completely covers face $ABCD$. (He is able to do this so that no water is lost!) At this point, the water still covers $\frac{4}{5}$ of the base area.

Determine the depth of the water, in centimetres, when the container is level.





Problem of the Week
Problem D and Solution
Tilt!

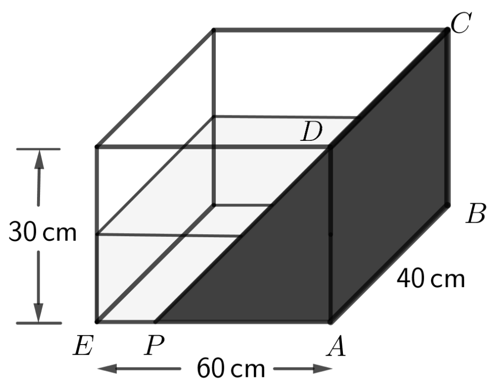
Problem

Troy has a container in the shape of a rectangular prism with a 40 cm by 60 cm base and a height of 30 cm. He labels the vertices of a 40 cm by 30 cm side face A, B, C, and D, with A and B being vertices of the base face too. He then puts some water in the container and tilts the container along AB until the water completely covers face $ABCD$. (He is able to do this so that no water is lost!) At this point, the water still covers $\frac{4}{5}$ of the base area.

Determine the depth of the water, in centimetres, when the container is level.

Solution

Let the other vertex on the bottom face adjacent to A be labelled E. Thus, $EA = 60$ cm. Let P be the point on EA such that $AP = \frac{4}{5}(EA) = \frac{4}{5}(60) = 48$ cm.



When the tank is tilted so that the water completely covers side face $ABCD$, a triangular prism with triangular base ADP and height 40 cm is created. Also, $\triangle ADP$ is a right-angled triangle, so when finding the area of $\triangle ADP$ we can use AP as the base of the triangle and AD as the height of the triangle. That is,

$$\begin{aligned}\text{Volume of triangular prism} &= \text{Area of } \triangle APD \times \text{height of triangular prism}\\ &= \frac{1}{2}(AP)(AD) \times (AB)\\ &= \frac{1}{2}(48)(30) \times (40)\\ &= 28\,800 \text{ cm}^3\end{aligned}$$

Let h represent the height of the water when the tank is level. The volume of the rectangular prism with base 40 cm by 60 cm and height h is the same as the volume of the triangular prism formed when the tank is tilted. That is,

$$\begin{aligned}60 \times 40 \times h &= 28\,800\\ 2400h &= 28\,800\\ h &= 12\end{aligned}$$

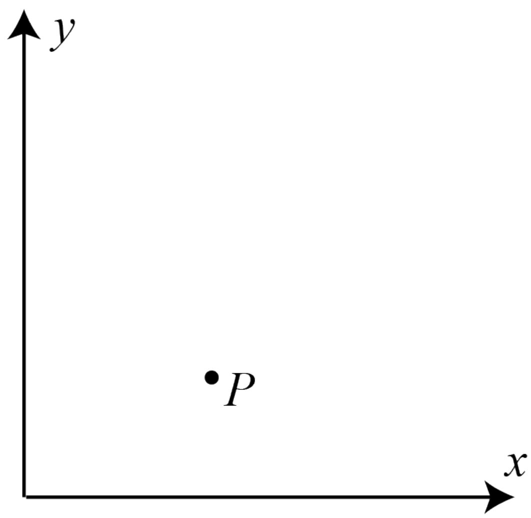
Therefore, the water is 12 cm deep when the container is level.

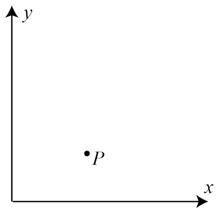
Problem of the Week

Problem D

Slope on a Plane

Percy drew x and y axes on grid paper and then plotted the point $P(8, 5)$. Quinlan then chose a different point, Q, and said its coordinates were each positive integers less than or equal to 20. Determine the probability that the slope of PQ is 0, 1, or 2.





Problem of the Week

Problem D and Solution

Slope on a Plane

Problem

Percy drew x and y axes on grid paper and then plotted the point $P(8, 5)$. Quinlan then chose a different point, Q, and said its coordinates were each positive integers less than or equal to 20. Determine the probability that the slope of PQ is 0, 1, or 2.

Solution

First, suppose the slope of PQ is 0. Then Q must have a y-coordinate of 5. Since the coordinates of Q are each positive integers less than or equal to 20, the smallest possible x-coordinate is 1 and the largest is 20. Thus, there are $20-1+1 = 20$ possible points. However, this includes the point P, so there are $20-1 = 19$ possibilities for Q such that PQ has slope 0.

Next, suppose the slope of PQ is 1. Let the coordinates of Q be (a, b). Then

$$\begin{aligned}\frac{b-5}{a-8} &= 1\\ b-5 &= a-8\\ b &= a-3\end{aligned}$$

Since a and b are each positive integers less than or equal to 20, the point with the smallest possible value for a is $(4, 1)$. Similarly, the point with the largest possible value for a is $(20, 17)$. Since $b = a - 3$, there is a possible value for b for each value of a between 4 and 20. Thus, there are $20-4+1 = 17$ possible points. However, this includes the point P, so there are $17-1 = 16$ possibilities for Q such that PQ has slope 1.

Finally, suppose the slope of PQ is 2. Let the coordinates of Q be (a, b). Then

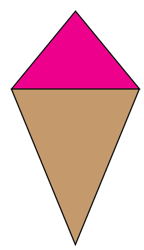
$$\begin{aligned}\frac{b-5}{a-8} &= 2\\ b-5 &= 2(a-8)\\ b-5 &= 2a-16\\ b &= 2a-11\end{aligned}$$

Since a and b are each positive integers less than or equal to 20, the point with the smallest possible value for a is $(6, 1)$. To determine the point with the largest possible value for a, we first notice that if $a = 20$, then $b > 20$. Then we can set $b = 2a - 11 < 20$. Thus $2a < 31$, or $a < 15.5$. It follows that the point with the largest possible value for a is $(15, 19)$. Since $b = 2a - 11$, there is a possible value for b for each value of a between 6 and 15. Thus, there are $15-6+1 = 10$ possible points. However, this includes the point P, so there are $10-1 = 9$ possibilities for Q such that PQ has slope 2.

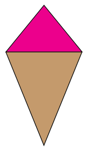
Thus, the total number of possibilities for Q such that PQ has slope 0, 1, or 2 is $19+16+9 = 44$. The total number of possibilities for Q is $20 \times 20 - 1 = 399$. Thus, the probability that the slope of PQ is 0, 1, or 2 is equal to $\frac{44}{399}$, or approximately 11%.

Problem of the Week
Problem D
Who Wants Ice Cream?

Xavier made a quilt block to represent an ice cream cone. The quilt block is composed of two isosceles triangles arranged to form a kite. The top triangle represents the ice cream and the bottom triangle represents the cone. The height of the bottom triangle is twice the height of the top triangle. The base of each triangle is $\frac{3}{4}$ of the height of the bottom triangle. If the area of the quilt block of the ice cream cone is 576 units^2, what is its perimeter?



Note: You may use the fact that the altitude of an isosceles triangle drawn to the unequal side bisects the unequal side.



Problem of the Week

Problem D and Solution

Who Wants Ice Cream?

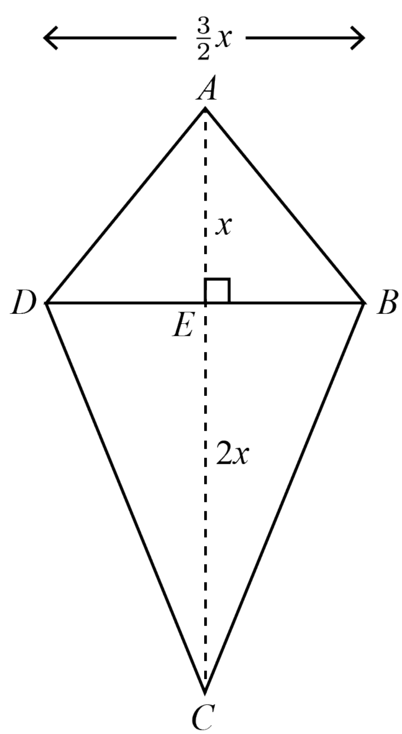
Problem

Xavier made a quilt block to represent an ice cream cone. The quilt block is composed of two isosceles triangles arranged to form a kite. The top triangle represents the ice cream and the bottom triangle represents the cone. The height of the bottom triangle is twice the height of the top triangle. The base of each triangle is $\frac{3}{4}$ of the height of the bottom triangle. If the area of the quilt block of the ice cream cone is 576 units^2, what is its perimeter?

NOTE: You may use the fact that the altitude of an isosceles triangle drawn to the unequal side bisects the unequal side.

Solution

We will call the top triangle $\triangle ABD$ and the bottom triangle $\triangle BCD$, having common base BD. Let E be on BD such that AE is an altitude of $\triangle ABD$. Then, since $\triangle ABD$ is isosceles, $BE = DE$. Notice that CE must also be an altitude of $\triangle BCD$, because if F is the point on BD such that CF is an altitude of $\triangle BCD$, then $BF = DF$. Since both E and F bisect BD, it must be the case that F and E represent the same point. Let $AE = x$. Then $CE = 2x$, and $BD = \left(\frac{3}{4}\right)(2x) = \frac{3}{2}x$.



We can use the area of $ABCD$ to determine the value of x.

$$\begin{aligned}\text{Area of } ABCD &= \text{Area of } \triangle ABD + \text{Area of } \triangle BCD\\ &= \left(\frac{1}{2}\right)(BD)(AE) + \left(\frac{1}{2}\right)(BD)(CE)\end{aligned}$$

Therefore,

$$\begin{aligned}576 &= \left(\frac{1}{2}\right)\left(\frac{3}{2}x\right)(x) + \left(\frac{1}{2}\right)\left(\frac{3}{2}x\right)(2x)\\ &= \frac{3x^2}{4} + \frac{6x^2}{4}\\ &= \frac{9x^2}{4}\\ x^2 &= \frac{576 \times 4}{9} = 256\end{aligned}$$

Thus, $x = 16$, since $x > 0$. Thus, $AE = 16$, $CE = 2x = 32$ and $BD = \frac{3}{2}x = 24$.

Now to find the perimeter of $ABCD$, we need to find the lengths of the sides. Since $\triangle ABD$ is isosceles, it follows that AE bisects BD. Thus, $DE = \frac{1}{2}BD = \frac{24}{2} = 12$. Using the Pythagorean Theorem in $\triangle AED$,

$$\begin{aligned}AD^2 &= AE^2 + DE^2\\ &= 16^2 + 12^2\\ &= 400\end{aligned}$$

Thus $AD = 20$, since $AD > 0$. Since $\triangle ABD$ is isosceles, $AB = AD = 20$.

Using the Pythagorean Theorem in $\triangle CED$,

$$\begin{aligned}CD^2 &= CE^2 + DE^2\\ &= 32^2 + 12^2\\ &= 1168\end{aligned}$$

Thus $CD = \sqrt{1168}$, since $CD > 0$. Since $\triangle BCD$ is isosceles, $BC = CD = \sqrt{1168}$.

Therefore, the perimeter of $ABCD$ equals $AD + AB + BC + CD = 20 + 20 + \sqrt{1168} + \sqrt{1168} = 40 + 2\sqrt{1168} \approx 108.35$.

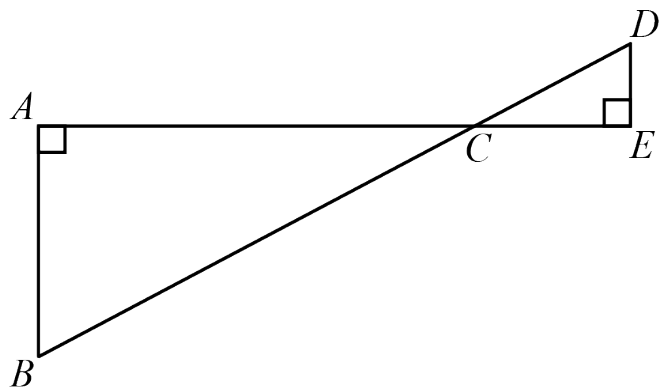
Note that we can simplify $\sqrt{1168}$ as follows:

$$\sqrt{1168} = \sqrt{16 \times 73} = 4\sqrt{73}$$

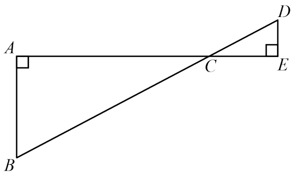
Therefore, the exact perimeter is $40 + 2\sqrt{1168} = 40 + 2(4\sqrt{73}) = 40 + 8\sqrt{73}$.

Problem of the Week
Problem D
Triangled

Consider the following diagram.



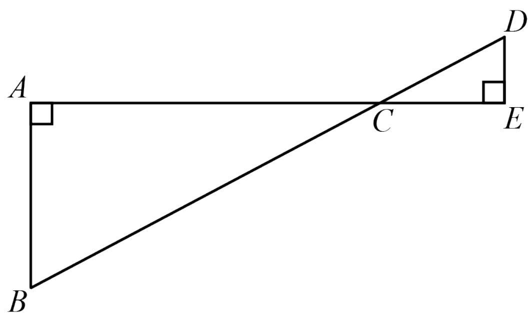
If $AE = 60$, $DE = 8$, and $CD = 17$, determine the length of BC.



Problem of the Week
Problem D and Solution
Triangled

Problem

Consider the following diagram.



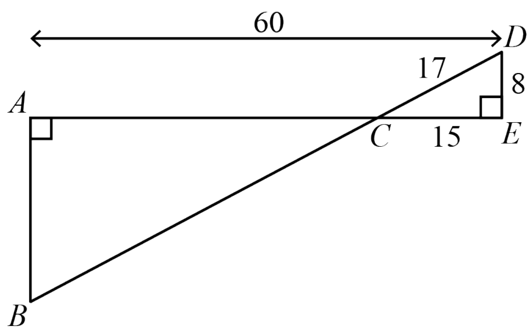
If $AE = 60$, $DE = 8$, and $CD = 17$, determine the length of BC.

Solution

Since $\triangle CDE$ is right-angled, we use the Pythagorean theorem to solve for CE.

$$CE^2 = CD^2 - DE^2 = 17^2 - 8^2 = 225$$

Thus $CE = \sqrt{225} = 15$, since $CE > 0$. The diagram is now updated with all the lengths we know so far.



Since $AC + CE = AE$, it follows that $AC = 60 - 15 = 45$. Since $\angle ACB$ and $\angle DCE$ are opposite angles, then $\angle ACB = \angle DCE$. We also know that $\angle CAB = \angle CED = 90°$, so we can conclude that $\triangle ABC \sim \triangle EDC$. Then,

$$\frac{BC}{AC} = \frac{CD}{CE}$$
$$\frac{BC}{45} = \frac{17}{15}$$
$$BC = 51$$

Thus, the length of BC is 51.

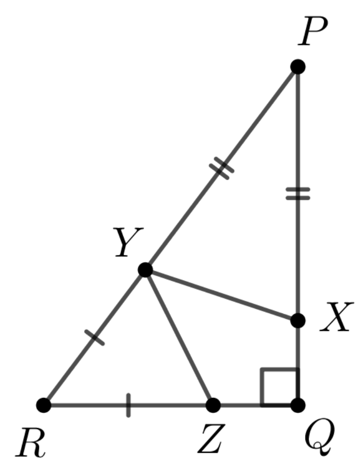
Problem of the Week

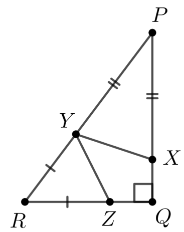
Problem D

Isosceles Delight

$\triangle PQR$ is right-angled at Q. Point X lies on PQ, point Z lies on QR, and point Y lies on PR such that $PX = PY$ and $RZ = RY$.

Determine the measure of $\angle XYZ$.





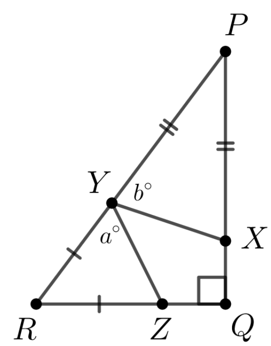
Problem of the Week
Problem D and Solution
Isosceles Delight

Problem

$\triangle PQR$ is right-angled at Q. Point X lies on PQ, point Z lies on QR, and point Y lies on PR such that $PX = PY$ and $RZ = RY$.

Determine the measure of $\angle XYZ$.

Solution
Solution 1



Let $\angle RYZ = a^\circ$ and $\angle PYX = b^\circ$.
Since $RY = RZ$, $\triangle RYZ$ is isosceles and $\angle RZY = \angle RYZ = a^\circ$.
Since $PY = PX$, $\triangle PYX$ is isosceles and $\angle PXY = \angle PYX = b^\circ$.

The angles in a triangle sum to 180°. Therefore, in $\triangle RYZ$,

$$\begin{aligned}\angle YRZ + \angle RYZ + \angle RZY &= 180^\circ\\ \angle YRZ + a^\circ + a^\circ &= 180^\circ\\ \angle YRZ &= 180^\circ - 2a^\circ\end{aligned}$$

In $\triangle PQR$,

$$\begin{aligned}\angle RPQ + \angle PQR + \angle QRP &= 180^\circ\\ \angle RPQ + 90^\circ + (180^\circ - 2a^\circ) &= 180^\circ\\ \angle RPQ &= 2a^\circ - 90^\circ\end{aligned}$$

In $\triangle PYX$,

$$\begin{aligned}\angle PYX + \angle PXY + \angle YPX &= 180^\circ\\ b^\circ + b^\circ + \angle YPX &= 180^\circ\end{aligned}$$

Since $\angle YPX = \angle RPQ$ (same angle), we have

$$\begin{aligned}b^\circ + b^\circ + (2a^\circ - 90^\circ) &= 180^\circ\\ 2b^\circ &= 270^\circ - 2a^\circ\\ b^\circ &= 135^\circ - a^\circ\end{aligned}$$

Now, PYR forms a straight line, so $\angle PYR = 180^\circ$. That is,

$$\begin{aligned}\angle PYX + \angle XYZ + \angle RYZ &= 180^\circ\\ b^\circ + \angle XYZ + a^\circ &= 180^\circ\\ (135^\circ - a^\circ) + \angle XYZ + a^\circ &= 180^\circ\\ \angle XYZ &= 180^\circ - 135^\circ = 45^\circ\end{aligned}$$

Therefore, $\angle XYZ = 45^\circ$. Note that in solving for $\angle XYZ$ it is was not necessary to determine either a° or b°.

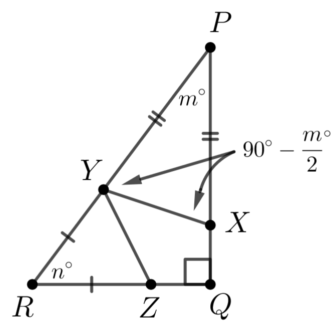
Solution 2

In $\triangle PQR$, let $\angle RPQ = m^\circ$ and $\angle PRQ = n^\circ$. The angles in a triangle sum to 180°. Therefore, in $\triangle PQR$,

$$\begin{aligned} \angle RPQ + \angle PQR + \angle PRQ &= 180^\circ \\ m^\circ + 90^\circ + n^\circ &= 180^\circ \\ m^\circ + n^\circ &= 90^\circ \end{aligned}$$

Therefore, $m + n = 90$.

Since $PY = PX$, $\triangle PYX$ is isosceles and so $\angle PXY = \angle PYX$.
Also, in $\triangle PYX$



$$\begin{aligned} \angle PYX + \angle PXY + \angle YPX &= 180^\circ \\ \angle PYX + \angle PYX + m^\circ &= 180^\circ \\ 2\angle PYX &= 180^\circ - m^\circ \end{aligned}$$

Therefore, $\angle PXY = \angle PYX = 90^\circ - \left(\frac{m}{2}\right)^\circ$.

Similarly, since $RY = RZ$, $\triangle RYZ$ is isosceles, and therefore $\angle RYZ = \angle RZY$. Also, in $\triangle RYZ$,

$$\begin{aligned} \angle RYZ + \angle RZY + \angle YRZ &= 180^\circ \\ \angle RYZ + \angle RYZ + n^\circ &= 180^\circ \\ 2\angle RYZ &= 180^\circ - n^\circ \end{aligned}$$

Therefore, $\angle RYZ = \angle RZY = 90^\circ - \left(\frac{n}{2}\right)^\circ$.

Since $\angle PYR = 180^\circ$, we have

$$\begin{aligned} \angle PYX + \angle XYZ + \angle RYZ &= 180^\circ \\ \left(90^\circ - \left(\frac{m}{2}\right)^\circ\right) + \angle XYZ + \left(90^\circ - \left(\frac{n}{2}\right)^\circ\right) &= 180^\circ \\ 180^\circ - \left(\frac{m}{2}\right)^\circ - \left(\frac{n}{2}\right)^\circ + \angle XYZ &= 180^\circ \\ \angle XYZ &= \left(\frac{m}{2}\right)^\circ + \left(\frac{n}{2}\right)^\circ \\ &= \left(\frac{m+n}{2}\right)^\circ \end{aligned}$$

Therefore, since $m + n = 90$, we have $\angle XYZ = \left(\frac{90}{2}\right)^\circ = 45^\circ$.

Therefore, $\angle XYZ = 45^\circ$. Note that in solving for $\angle XYZ$ it is was not necessary to determine either m° or n°.

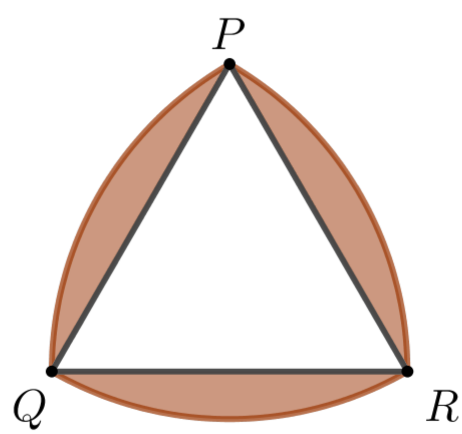
Problem of the Week

Problem D

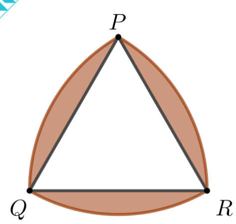
Painting a Logo

Nathaniel has designed a new logo for his school's math club. When drawing his logo, he starts with an equilateral triangle, labelled $\triangle PQR$, with sides of length 20 cm. He then draws in minor arc PQ, which is an arc of the circle with centre R and radius RQ, followed by minor arc PR, which is an arc of the circle with centre Q and radius QP, and then minor arc RQ, which is an arc of a circle with centre P and radius PR.

He wants to colour the region bounded by each arc but outside of $\triangle PQR$. Determine the total area to be coloured, correct to one decimal place.



Note: You may use the fact that an altitude in an equilateral triangle bisects the side it is drawn to.



Problem of the Week

Problem D and Solution

Painting a Logo

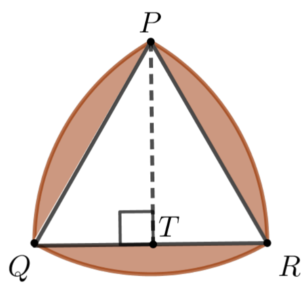
Problem

Nathaniel has designed a new logo for his school's math club. When drawing his logo, he starts with an equilateral triangle, labelled $\triangle PQR$, with sides of length 20 cm. He then draws in minor arc PQ, which is an arc of the circle with centre R and radius RQ, followed by minor arc PR, which is an arc of the circle with centre Q and radius QP, and then minor arc RQ, which is an arc of a circle with centre P and radius PR.

He wants to colour the region bounded by each arc but outside of $\triangle PQR$. Determine the total area to be coloured, correct to one decimal place.

Solution

We'll first determine the area of $\triangle PQR$. Construct altitude PT. Since $\triangle PQR$ is equilateral, it follows that PT bisects QR.



Since $QR = 20$ cm, it follows that $TR = 10$ cm. By the Pythagorean Theorem, $PR^2 = TR^2 + PT^2$. Therefore, $20^2 = 10^2 + PT^2$, and $PT^2 = 400 - 100 = 300$ follows. Since $PT > 0$, we have $PT = \sqrt{300}$ cm.

Therefore, the area of $\triangle PQR$ is $\dfrac{(QR) \times (PT)}{2} = \dfrac{20 \times \sqrt{300}}{2} = 10\sqrt{300}$ cm^2.

The logo consists of three overlapping circle sectors, one with centre P, one with centre Q, and one with centre R. Each circle sector has the same radius, 20 cm, and a 60° central angle. Therefore, each sector has the same area, which is $60 \div 360$, or one-sixth, the area of a circle of radius 20 cm.

That is, the area of each sector is equal to $\frac{1}{6}\pi r^2 = \frac{1}{6}\pi(20)^2 = \frac{200}{3}\pi$ cm^2.

The coloured part of each circle sector is equal to the area of the sector minus the area of $\triangle PQR$. That is, it is equal to

$$\left(\frac{200}{3}\pi - 10\sqrt{300}\right) \text{ cm}^2$$

Since there are three congruent coloured areas, the total area to be coloured is equal to

$$3 \times \left(\frac{200}{3}\pi - 10\sqrt{300}\right) = 200\pi - 30\sqrt{300} \approx 108.7 \text{ cm}^2$$

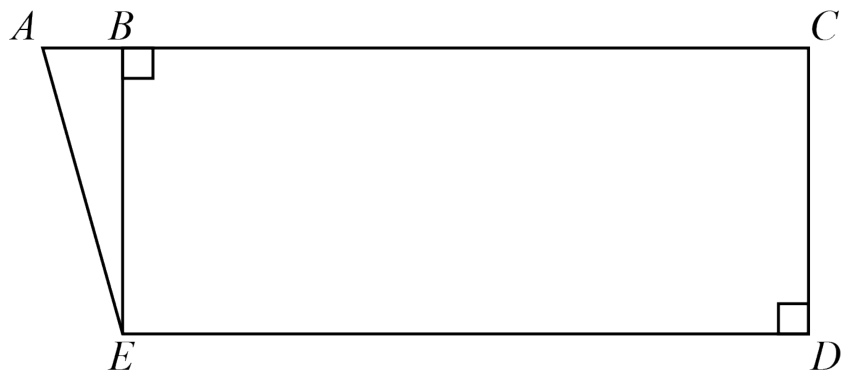
Problem of the Week

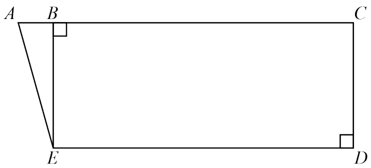
Problem D

A Triangle and a Rectangle

Point B lies on line segment AC, in between A and C, so that $BC = 10 \times AB$. Line segment DE is parallel to AC so that $BCDE$ forms a rectangle and ABE forms a right-angled triangle.

If $AE = 25$ and $BD = 74$, determine the exact value of the length of AD.





Problem of the Week

Problem D and Solution

A Triangle and a Rectangle

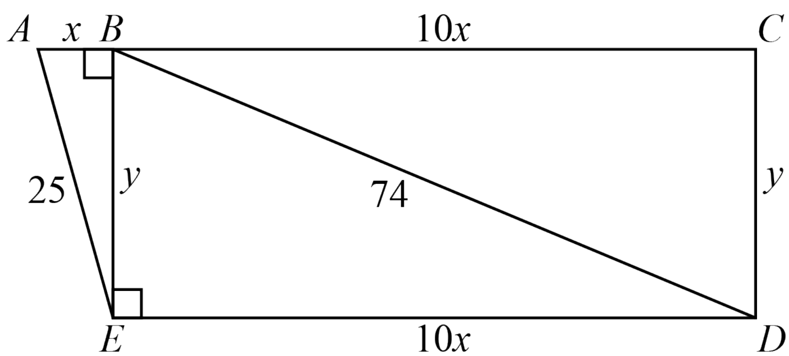
Problem

Point B lies on line segment AC, in between A and C, so that $BC = 10 \times AB$. Line segment DE is parallel to AC so that $BCDE$ forms a rectangle and ABE forms a right-angled triangle.

If $AE = 25$ and $BD = 74$, determine the exact value of the length of AD.

Solution

Let $AB = x$. Then $BC = 10 \times AB = 10x$. Let $CD = y$. Then since $BCDE$ is a rectangle, it follows that $BE = CD = y$ and $ED = BC = 10x$. We label our diagram accordingly.



Using the Pythagorean Theorem in $\triangle ABE$,

$$\begin{aligned} AB^2 + BE^2 &= AE^2 \\ x^2 + y^2 &= 25^2 \\ x^2 + y^2 &= 625 \\ y^2 &= 625 - x^2 \end{aligned}$$

Since $\angle BED = 90^\circ$, it follows that $\triangle BED$ is a right-angled triangle. Using the Pythagorean Theorem in $\triangle BED$,

$$\begin{aligned} BE^2 + ED^2 &= BD^2 \\ y^2 + (10x)^2 &= 74^2 \\ y^2 + 100x^2 &= 5476 \\ y^2 &= 5476 - 100x^2 \end{aligned}$$

Then, since $y^2 = 625 - x^2$ and $y^2 = 5476 - 100x^2$, it follows that

$$\begin{aligned} 625 - x^2 &= 5476 - 100x^2 \\ 99x^2 &= 4851 \\ x^2 &= 49 \end{aligned}$$

Thus, $x = \sqrt{49} = 7$, since $x > 0$.

Then $y^2 = 625 - x^2 = 625 - (7)^2 = 576$. Thus, $y = \sqrt{576} = 24$, since $y > 0$.

It follows that $CD = y = 24$ and
$AC = AB + BC = x + 10x = 11x = 11(7) = 77$.

Since $\angle ACD = 90°$, it follows that $\triangle ACD$ is a right-angled triangle. Using the Pythagorean Theorem in $\triangle ACD$,

$$\begin{aligned} AC^2 + CD^2 &= AD^2 \\ 77^2 + 24^2 &= AD^2 \\ 6505 &= AD^2 \end{aligned}$$

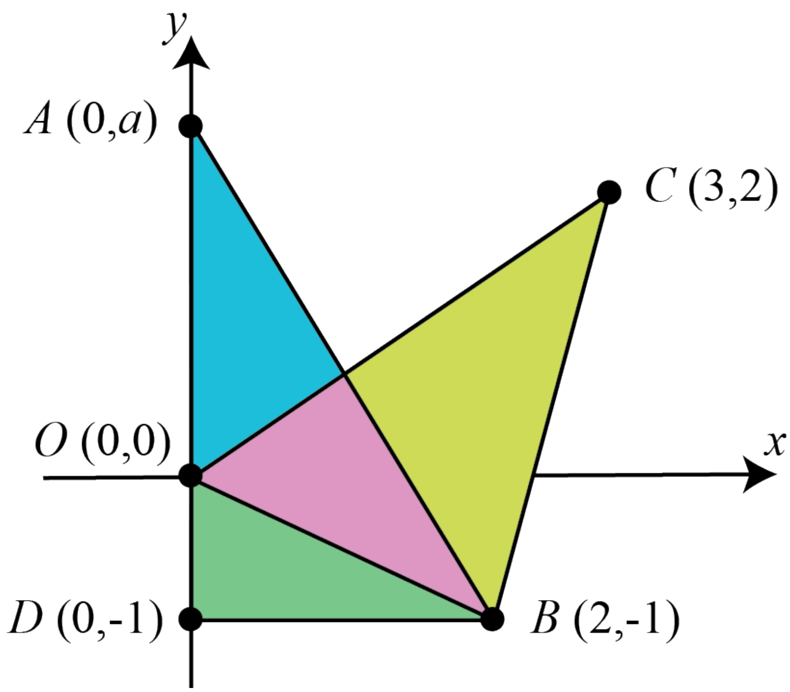
Thus, since $AD > 0$, we can conclude that $AD = \sqrt{6505}$.

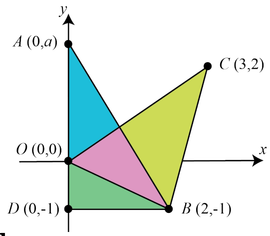
Problem of the Week

Problem D

Stained Glass

Points $A(0, a)$, $B(2, -1)$, $C(3, 2)$, $D(0, -1)$, and $O(0, 0)$ are such that $\triangle ABD$ and $\triangle COB$ have the same area. If $a > 0$, determine the value of a.





Problem of the Week
Problem D and Solution
Stained Glass

Problem

Points $A(0,a)$, $B(2,-1)$, $C(3,2)$, $D(0,-1)$, and $O(0,0)$ are such that $\triangle ABD$ and $\triangle COB$ have the same area. If $a > 0$, determine the value of a.

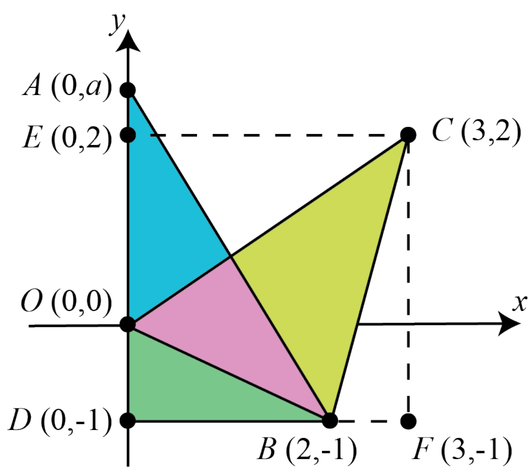
Solution

Solution 1

In $\triangle ABD$, $AD = a - (-1) = a + 1$ and $DB = 2 - 0 = 2$.

Thus, area $\triangle ABD = \frac{AD \times DB}{2} = \frac{(a+1) \times 2}{2} = a + 1$ units^2.

To determine the area of $\triangle COB$, we consider points $E(0,2)$ and $F(3,-1)$ and draw in $ECFD$.



Since E and D both have x-coordinate 0, ED is a vertical line which passes through O. Since C and F have the same x-coordinate, CF is also a vertical line. Since E and C have the same y-coordinate, EC is a horizontal line. Since D and F both have y-coordinate -1, DF is also a horizontal line which passes through B. Thus, $ECFD$ is a rectangle that encloses $\triangle COB$, and we have

$$\text{area } \triangle COB = \text{area } ECFD - \text{area } \triangle CEO - \text{area } \triangle ODB - \text{area } \triangle BFC$$

In rectangle $ECFD$, $EC = 3 - 0 = 3$ and $ED = 2 - (-1) = 3$. The area of rectangle $EDFC = EC \times ED = 3 \times 3 = 9$ units^2.

Since $ECFD$ is a rectangle, $\triangle CEO$ is right-angled at E. Since $EC = 3$ and $EO = 2 - 0 = 2$, the area of $\triangle CEO = \frac{EC \times EO}{2} = \frac{3 \times 2}{2} = 3$ units^2.

Since $ECFD$ is a rectangle, $\triangle ODB$ is right-angled at D. Since $OD = 0 - (-1) = 1$ and $DB = 2 - 0 = 2$, the area of $\triangle ODB = \frac{OD \times DB}{2} = \frac{1 \times 2}{2} = 1$ unit^2.

Since $ECFD$ is a rectangle, $\triangle BFC$ is right-angled at F. Since $BF = 3 - 2 = 1$ and $CF = 2 - (-1) = 3$, the area of $\triangle BFC = \frac{BF \times CF}{2} = \frac{1 \times 3}{2} = 1.5$ units^2.

Thus,
$$\begin{aligned}\text{area } \triangle COB &= \text{area } ECFD - \text{area } \triangle CEO - \text{area } \triangle ODB - \text{area } \triangle BFC\\ &= 9 - 3 - 1 - 1.5\\ &= 3.5 \text{ units}^2\end{aligned}$$

We're given that $\triangle ABD$ and $\triangle COB$ have the same area. Thus, the area of $\triangle ABD = 3.5$ units^2.

Since the area of $\triangle ABD = a + 1$ units^2, we have $a + 1 = 3.5$ and $a = 2.5$ follows.

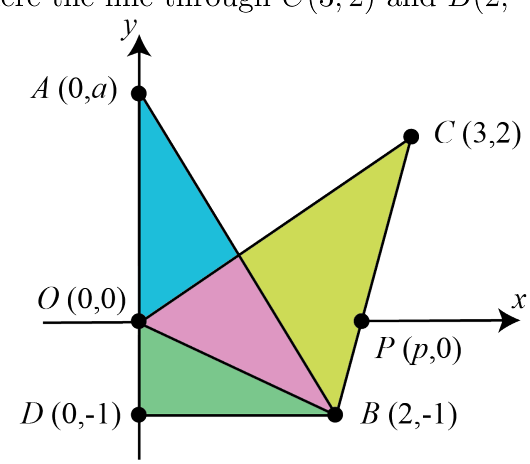
Therefore, the value of a is 2.5.

Solution 2

In $\triangle ABD$, $AD = a - (-1) = a + 1$ and $DB = 2 - 0 = 2$.

Thus, area $\triangle ABD = \frac{AD \times DB}{2} = \frac{(a+1)\times 2}{2} = a + 1$ units^2.

Let $P(p, 0)$ be the point where the line through $C(3, 2)$ and $B(2, -1)$ intersects the x-axis.



We have area $\triangle COB$ = area $\triangle COP$ + area $\triangle BOP$.

To determine the value of p, we first determine the equation of the line through $C(3, 2)$ and $B(2, -1)$.

Since the slope of the line is $\frac{2-(-1)}{3-2} = 3$, the equation of the line is of the form $y = 3x + b$, for some b. Substituting $x = 3$ and $y = 2$ gives $2 = 3(3) + b$ and $b = -7$ follows. Therefore, the equation of the line though $C(3, 2)$ and $B(2, -1)$ is $y = 3x - 7$.

Substituting $x = p$ and $y = 0$ into $y = 3x - 7$ we obtain $0 = 3p - 7$ and $p = \frac{7}{3}$ follows.

In $\triangle COP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the x-axis to $C(3, 2)$, which is 2 units. Therefore, the area of $\triangle COP = \frac{\frac{7}{3}\times 2}{2} = \frac{7}{3}$ units^2.

In $\triangle BOP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the x-axis to $B(2, -1)$, which is 1 unit. Therefore, the area of $\triangle BOP = \frac{\frac{7}{3}\times 1}{2} = \frac{7}{6}$ units^2.

Therefore, area $\triangle COB$ = area $\triangle COP$ + area $\triangle BOP = \frac{7}{3} + \frac{7}{6} = \frac{14}{6} + \frac{7}{6} = \frac{21}{6} = \frac{7}{2}$ units^2.

We're given that $\triangle ABD$ and $\triangle COB$ have the same area. Thus, the area of $\triangle ABD = \frac{7}{2}$ units^2.

Since the area of $\triangle ABD = a + 1$ units^2, we have $a + 1 = \frac{7}{2}$ and $a = \frac{5}{2} = 2.5$ follows.

Therefore, the value of a is 2.5.

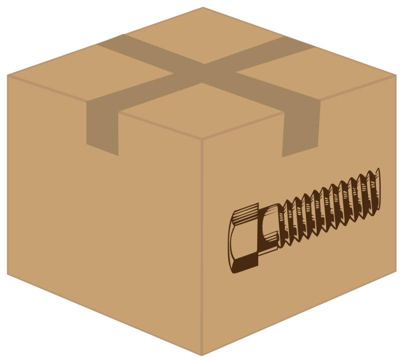
Number Sense (N)

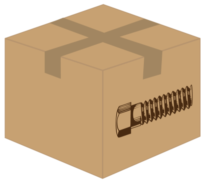
Take me to the cover

Problem of the Week
Problem D
More or Less

Severin works in a factory that produces steel bolts. One type of steel bolt should have a mass of 5 g; however, there is some variation, so each bolt may be heavier or lighter by as much as 1.9%. Severin puts together a box of these bolts and notices that the total mass of the bolts in the box is exactly 1 kg.

Determine the minimum and maximum number of bolts that could be in the box.





Problem of the Week
Problem D and Solution
More or Less

Problem

Severin works in a factory that produces steel bolts. One type of steel bolt should have a mass of 5 g; however, there is some variation, so each bolt may be heavier or lighter by as much as 1.9%. Severin puts together a box of these bolts and notices that the total mass of the bolts in the box is exactly 1 kg.

Determine the minimum and maximum number of bolts that could be in the box.

Solution

In order to find the minimum number of bolts, we consider the case where each bolt has the largest possible mass. Since the mass of one bolt can be at most 1.9% more than 5 g, the largest possible mass of one bolt is $5 \times 1.019 = 5.095$ g. If every bolt in the box had this mass, then the total number of bolts in the box would be $1000 \div 5.095 \approx 196.27$. However, the number of bolts must be an integer, so the minimum number of bolts is therefore 197. If there were only 196 bolts, then their total mass would be $196 \times 5.095 = 998.62$ g, which is less than 1 kg.

Similarly, in order to find the maximum number of bolts, we consider the case where each bolt has the smallest possible mass. Since the mass of one bolt can be at most 1.9% less than 5 g, the smallest possible mass of one bolt is $5 \times (1 - 0.019) = 5 \times 0.981 = 4.905$ g. If every bolt in the box had this mass, then the total number of bolts in the box would be $1000 \div 4.905 \approx 203.87$. However, the number of bolts must be an integer, so the maximum number of bolts is therefore 203. If there were 204 bolts, then their total mass would be $204 \times 4.905 = 1000.62$ g, which is more than 1 kg.

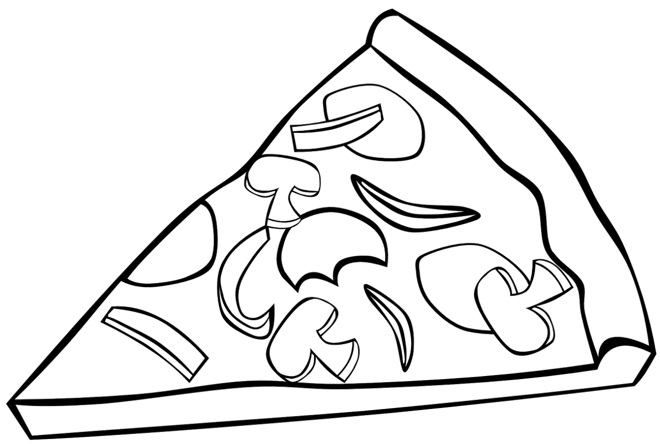
Therefore, the minimum number of bolts is 197 and the maximum number of bolts is 203.

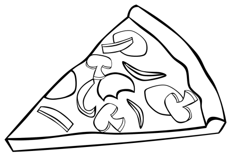
Problem of the Week

Problem D

Ten Slice Pizza

The DECI-Pizza Company has a special pizza that has 10 slices. Two of the slices are each $\frac{1}{6}$ of the whole pizza, two are each $\frac{1}{8}$, four are each $\frac{1}{12}$, and two are each $\frac{1}{24}$. A group of n friends share the pizza by distributing all of these slices. They do not cut any of the slices. Each of the n friends receives, in total, an equal fraction of the whole pizza. For what values of $n > 1$ is this possible?





Problem of the Week
Problem D and Solution
Ten Slice Pizza

Problem

The DECI-Pizza Company has a special pizza that has 10 slices. Two of the slices are each $\frac{1}{6}$ of the whole pizza, two are each $\frac{1}{8}$, four are each $\frac{1}{12}$, and two are each $\frac{1}{24}$. A group of n friends share the pizza by distributing all of these slices. They do not cut any of the slices. Each of the n friends receives, in total, an equal fraction of the whole pizza. For what values of $n > 1$ is this possible?

Solution

Solution 1

Each of the n friends is to receive $\frac{1}{n}$ of the pizza.

Since there are two slices that are each $\frac{1}{6}$ of the pizza and these slices cannot be cut, then each friend receives at least $\frac{1}{6}$ of the pizza. This means that there cannot be more than 6 friends. That is, $n \leq 6$.

The value $n = 2$ is possible. We show this by dividing the slices into two groups, each of which totals $\frac{1}{2}$ of the pizza. Note that $\frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2}$. This means the other six slices must also add to $\frac{1}{2}$.

The value $n = 3$ is possible. We show this by finding three groups of slices, with each group totaling $\frac{1}{3}$ of the pizza. Since $2 \times \frac{1}{6} = \frac{1}{3}$ and $4 \times \frac{1}{12} = \frac{1}{3}$, then the other four slices must also add to $\frac{1}{3}$ (the rest of the pizza), and so $n = 3$ is possible.

The value $n = 4$ is possible since $2 \times \frac{1}{8} = \frac{1}{4}$ and $\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$ (which can be done twice). The other four slices must also add to $\frac{1}{4}$.

The value $n = 6$ is possible since two slices are $\frac{1}{6}$ on their own, two groups of size $\frac{1}{6}$ can be made from the four slices of size $\frac{1}{12}$, and $\frac{1}{8} + \frac{1}{24} = \frac{1}{6}$ (which can be done twice), which makes six groups of size $\frac{1}{6}$.

The value $n = 5$ is not possible, since to make a portion of size $\frac{1}{5}$ that includes a slice of size $\frac{1}{6}$, the remaining slices must total $\frac{1}{5} - \frac{1}{6} = \frac{1}{30}$. Since every slice is larger than $\frac{1}{30}$, this is not possible.

Therefore, the possible values of n are 2, 3, 4, and 6.

Solution 2

The pizza is cut into two slices of size $\frac{1}{24}$, four slices of size $\frac{1}{12}$, four slices of size $\frac{1}{8}$, and two slices of size $\frac{1}{6}$.

Each of these fractions can be written with a denominator of 24. Thus, this is equivalent to saying there are two slices of size $\frac{1}{24}$, four slices of size $\frac{2}{24}$, two slices of size $\frac{3}{24}$, and two slices of size $\frac{4}{24}$.

To create groups of slices of equal total size, we can now consider combining the integers 1, 1, 2, 2, 2, 2, 3, 3, 4, and 4 into groups with equal sum. (These integers represent the size of each slice measured in units of $\frac{1}{24}$ of the pizza.)

Since the largest integer in the list is 4, then each group has to have size at least 4. Since $4 = 24 \div 6$, then the slices cannot be broken into more than 6 groups of equal size, which means that n cannot be greater than 6.

Here is a way of breaking the slices into $n = 6$ equal groups, each with total size $24 \div 6 = 4$:

$$4 \qquad 4 \qquad 3+1 \qquad 3+1 \qquad 2+2 \qquad 2+2$$

Here is a way of breaking the slices into $n = 4$ equal groups, each with total size $24 \div 4 = 6$:

$$4+2 \qquad 4+2 \qquad 3+3 \qquad 2+2+1+1$$

Here is a way of breaking the slices into $n = 3$ equal groups, each with total size $24 \div 3 = 8$:

$$4+4 \qquad 2+2+2+2 \qquad 3+3+1+1$$

Here is a way of breaking the slices into $n = 2$ equal groups, each with total size $24 \div 2 = 12$:

$$4+4+2+2 \qquad 3+3+2+2+1+1$$

Since 24 is not a multiple of 5, the slices cannot be broken into $n = 5$ groups of equal size.

Therefore, the possible values of n are 2, 3, 4, and 6.

Problem of the Week
Problem D
Summing up a Sequence 1

The first term in a sequence is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

Elias writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. What is the smallest possible value of n?

24, 12, 6, ...

24, 12, 6, ...

Problem of the Week
Problem D and Solution
Summing up a Sequence 1

Problem

The first term in a sequence is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

Elias writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. What is the smallest possible value of n?

Solution

We will begin by finding more terms in the sequence. The first 14 terms of the sequence are 24, 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1.

If we continue the sequence, we will see that the terms 4, 2, and 1 will continue to repeat. Now we want to find the smallest possible value of n so that the sum of the terms in the sequence from term 1 to term n is at least 1000.

The sum of the first 8 terms is $24 + 12 + 6 + 3 + 10 + 5 + 16 + 8 = 84$. The sum of the repeating numbers is $4 + 2 + 1 = 7$. We want to determine the number of groups of repeating numbers. Let this be g. Suppose $84 + 7g = 1000$. Solving this gives $7g = 916$, so $g \approx 130.857$.

If $g = 130$, then the sum of the terms in the sequence is $84 + 7 \times 130 = 994$. This sequence contains the first 8 terms, plus 130 groups of the three repeating numbers. Therefore there are a total of $8 + 3 \times 130 = 398$ terms.

The 399^{th} term in the sequence will be 4, so the sum of the first 399 terms will be $994 + 4 = 998$.

The 400^{th} term in the sequence will be 2, so the sum of the first 400 terms will be $998 + 2 = 1000$. This is the smallest possible four-digit number, so the smallest possible value of n is 400.

Extension:

In 1937, the mathematician Lothat Collatz wondered if any sequence whose terms after the first are determined in this way would always eventually reach the number 1, regardless of which number you started with. This problem is actually still unsolved today and is called the Collatz Conjecture.

Problem of the Week
Problem D
Student to Student

At the beginning of the school year, the ratio of the number of Grade 10 students to the number of Grade 9 students at CEMC H.S. was $15 : 16$. By the end of the year, there were 30 more Grade 10 students and there were 20 fewer Grade 9 students, and the ratio of the number of Grade 10 students to the number of Grade 9 students was now $11 : 10$.

How many Grade 9 students and how many Grade 10 students were there at the beginning of the school year?

Problem of the Week

Problem D and Solution

Student to Student

Problem

At the beginning of the school year, the ratio of the number of Grade 10 students to the number of Grade 9 students at CEMC H.S. was 15 : 16. By the end of the year, there were 30 more Grade 10 students and there were 20 fewer Grade 9 students, and the ratio of the number of Grade 10 students to the number of Grade 9 students was now 11 : 10.

How many Grade 9 students and how many Grade 10 students were there at the beginning of the school year?

Solution

Originally, the ratio of the number of Grade 10 students to the number of Grade 9 students was 15 : 16. Therefore, we can let the number of Grade 10 students at the beginning of the year be $15n$ and the number of Grade 9 students at the beginning of the year be $16n$, for some integer n.

Thus, at the end of the year, there were $15n + 30$ Grade 10 students and $16n - 20$ Grade 9 students.

Since the ratio of the number of Grade 10 students at the end of the year to the number of Grade 9 students at the end of the year is 11 : 10, we have

$$\begin{aligned} \frac{15n+30}{16n-20} &= \frac{11}{10} \\ 150n + 300 &= 176n - 220 \\ 520 &= 26n \\ n &= 20 \end{aligned}$$

Therefore, there were $16n = 16(20) = 320$ Grade 9 students and $15n = 15(20) = 300$ Grade 10 students at the beginning of the school year.

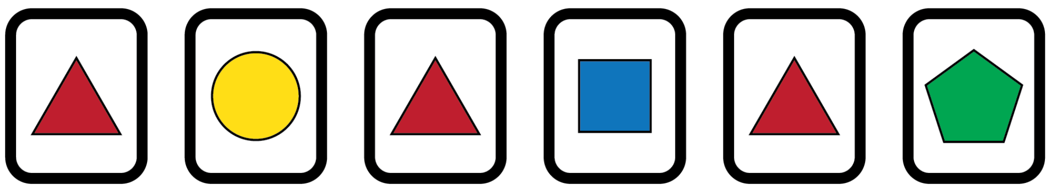
Problem of the Week

Problem D

Six Cards

Antonia has a set of cards where each card has a shape on one side and a digit from 0 to 9 on the other side. Any two cards with the same shape have the same digit on the other side, and any two cards with different shapes have different digits on the other side.

Antonia lays out the following six cards.



She then flips each card over in place and records the six-digit number they form. For example, if there is a 4 on the other side of the cards with a triangle, a 2 on the other side of the card with a circle, a 7 on the other side of the card with a square, and a 5 on the other side of the card with a pentagon, then the six-digit number they form would be 424 745.

Antonia notices that the six-digit number they form is divisible by 11. Determine the largest and smallest possible six-digit numbers that this could be.

Note: You may find the following fact useful:

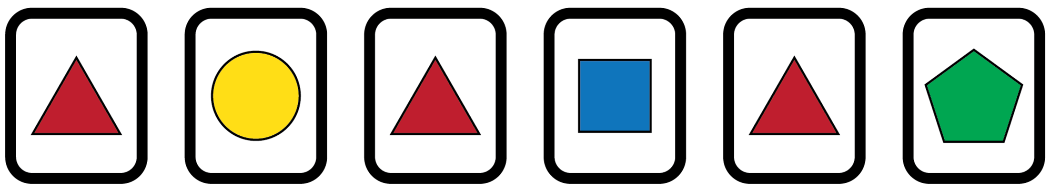
A number is divisible by 11 exactly when the sum of the digits in the odd digit positions minus the sum of the digits in the even digit positions is divisible by 11.
For example, the number 138 248 is divisible by 11 since
$(1 + 8 + 4) - (3 + 2 + 8) = 13 - 13 = 0$ and 0 is divisible by 11.
The number 693 748 is also divisible by 11 since
$(6 + 3 + 4) - (9 + 7 + 8) = 13 - 24 = -11$ and -11 is divisible by 11.

Problem of the Week
Problem D and Solution
Six Cards

Problem

Antonia has a set of cards where each card has a shape on one side and a digit from 0 to 9 on the other side. Any two cards with the same shape have the same digit on the other side, and any two cards with different shapes have different digits on the other side.

Antonia lays out the following six cards.



She then flips each card over in place and records the six-digit number they form. For example, if there is a 4 on the other side of the cards with a triangle, a 2 on the other side of the card with a circle, a 7 on the other side of the card with a square, and a 5 on the other side of the card with a pentagon, then the six-digit number they form would be 424 745.

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Solution

Let T represent the digit on the other side of the cards with the triangle, let C represent the digit on the other side of the card with the circle, let S represent the digit on the other side of the card with the square, and let P represent the digit on the other side of the card with the pentagon. Then the six-digit number can be represented as $TCT\,STP$.

We will start by finding the largest possible six-digit number. To do this, we will make the leftmost digit as large as possible. So $T = 9$. Then our six-digit number is $9C9\,S9P$. Next we will make C as large as possible, so $C = 8$. Then our six-digit number is $989\,S9P$. Next we will make S as large as possible, so $S = 7$. Then our six-digit number is $989\,79P$.

If $989\,79P$ is divisible by 11, then $(9+9+9)-(8+7+P) = 27-15-P = 12-P$ is also divisible by 11. The only possible single-digit value for P is $P=1$, and we have not already used this digit. Since $989\,791 = 11 \times 89\,981$, we can verify that $989\,791$ is divisible by 11. Thus, the largest possible six-digit number is $989\,791$.

Next we will find the smallest possible six-digit number. To do this, we will make the leftmost digit as small as possible. So $T=1$. Note that it's not possible for T to equal 0, because we need to have a six-digit number. Then our six-digit number is $1C1\,S1P$. Next we will make C as small as possible, so $C=0$. Then our six-digit number is $101\,S1P$. Next we will make S as small as possible, so $S=2$. Then our six-digit number is $101\,21P$.

If $101\,21P$ is divisible by 11, then $(1+1+1)-(0+2+P) = 3-2-P = 1-P$ is also divisible by 11. The only possible single-digit value for P is $P=1$, however we already set $T=1$. So we must try a larger value for S.

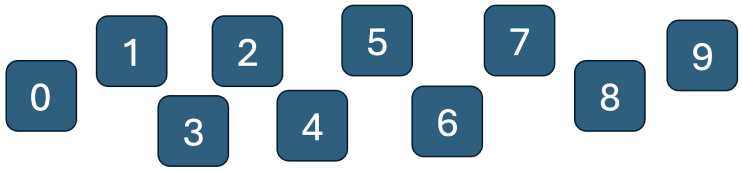
We will try the next smallest possible value for S, $S=3$. Then our six-digit number is $101\,31P$. If $101\,31P$ is divisible by 11, then $(1+1+1)-(0+3+P) = 3-3-P = -P$ is also divisible by 11. The only possible single-digit value for P is $P=0$, however we already set $C=0$. So we must try a larger value for S.

We will try $S=4$. Then our six-digit number is $101\,41P$. If $101\,41P$ is divisible by 11, then $(1+1+1)-(0+4+P) = 3-4-P = -1-P$ is also divisible by 11. There is no possible value for P that is a single digit, so we must try a larger value for S.

We will try $S=5$. Then our six-digit number is $101\,51P$. If $101\,51P$ is divisible by 11, then $(1+1+1)-(0+5+P) = 3-5-P = -2-P$ is also divisible by 11. The only possible single-digit value for P is $P=9$, and we have not already used this digit. Since $101\,519 = 11 \times 9229$, we can verify that $101\,519$ is divisible by 11. Thus, the smallest possible six-digit number is $101\,519$.

Problem of the Week
Problem D
Counting with Digit Tiles

For each digit from 0 to 9, you have 100 identical tiles with that digit on it.



Using these 1000 tiles, you start building consecutive positive integers, starting with 1. Each time you build an integer, you must remove the digit tiles you used from your stockpile of tiles.

For example, after you have built the integers from 1 to 13, the table below summarizes how many tiles remain for each digit.

Digit	0	1	2	3	4	5	6	7	8	9
Number of Tiles Remaining	99	94	98	98	99	99	99	99	99	99

What is the largest integer you can build without running out of the tiles needed to form the integer?

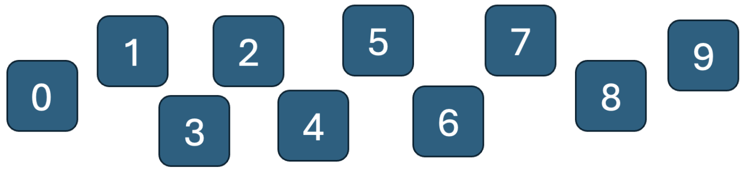
Problem of the Week

Problem D and Solution

Counting with Digit Tiles

Problem

For each digit from 0 to 9, you have 100 identical tiles with that digit on it.



Using these 1000 tiles, you start building consecutive positive integers, starting with 1. Each time you build an integer, you must remove the digit tiles you used from your stockpile of tiles.

For example, after you have built the integers from 1 to 13, the table below summarizes how many tiles remain for each digit.

Digit	0	1	2	3	4	5	6	7	8	9
Number of Tiles Remaining	99	94	98	98	99	99	99	99	99	99

What is the largest integer you can build without running out of the tiles needed to form the integer?

Solution

For the integers 1 to 99, each digit from 1 to 9 will be used the same number of times. The digit 0 will be used less than every other digit. As soon as we start to build the integer 100, the digit 1 will be used more frequently than any other digit. So, we will first run out of tiles with the digit 1 on them. Thus, we will count the number of times we can use a tile with the digit 1.

From 1 to 99, there is a 1 in the units digit of integers 1, 11, 21, 31, 41, 51, 61, 71, 81, 91, a total of 10 integers. There are also 10 integers with tens digit 1: 10, 11, 12, 13, 14, 15, 16, 17, 18, and 19. Thus, by the time we reach 99, we have used $10 + 10 = 20$ tiles with the digit 1.

From 100 to 109, we use the digit 1 in the hundreds position 10 times, and in the units position of integer 101. Thus, we use 11 more tiles with the digit 1, so we have now used $20 + 11 = 31$ such tiles.

From 110 to 119, we use the digit 1 in the hundreds position 10 times, in the tens position 10 times, and once in the units position. This makes a total of 21 more tiles with the digit 1 used, so we have now used $31 + 21 = 52$ such tiles.

For each of 120 to 129, 130 to 139, 140 to 149, and 150 to 159, we use the same number of tiles with the digit 1 as we did when building integers from 100 to 109. So, to build the integers from 120 to 159 we use $11 + 11 + 11 + 11 = 44$ tiles with the digit 1, and have now used a total of $52 + 44 = 96$ such tiles.

Now we have four tiles with the digit 1 remaining. We can build 160, 161, and 162. After we have built the integer 162, no more tiles with the digit 1 remain, and so we cannot build the integer 163.

Therefore, the largest integer we can build to before running out of the necessary digits to create the integer is 162.

Problem of the Week

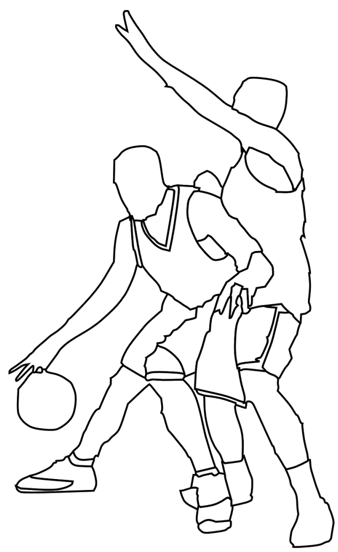
Problem D

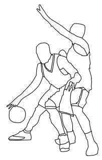
Height Correction

At a local high school, the basketball coach measured the height of each player. Nine of the ten heights, in cm, were 180, 181, 183, 187, 188, 190, 193, 195, and 196. The height of the tenth player, Richard, was a whole number.

Initially, the coach measured Richard's height incorrectly. After the error was corrected, both the mean and median of the heights increased by 0.5 cm.

Determine all the possible values for the height of Richard.





Problem of the Week

Problem D and Solution

Height Correction

Problem

At a local high school, the basketball coach measured the height of each player. Nine of the ten heights, in cm, were 180, 181, 183, 187, 188, 190, 193, 195, and 196. The height of the tenth player, Richard, was a whole number.

Initially, the coach measured Richard's height incorrectly. After the error was corrected, both the mean and median of the heights increased by 0.5 cm.

Determine all the possible values for the height of Richard.

Solution

We will let Richard's correct height be R and the incorrectly recorded height be r.

To find a relationship between R and r we will look at the averages of the heights. The average of the heights with the incorrect height, r, is

$$\frac{180+181+183+187+188+190+193+195+196+r}{10}=\frac{1693+r}{10}$$

The average with the correct height, R, is

$$\frac{180+181+183+187+188+190+193+195+196+R}{10}=\frac{1693+R}{10}$$

Since the new mean is 0.5 cm more than the old mean, we have:

$$\begin{aligned}\frac{1693+r}{10}+0.5&=\frac{1693+R}{10}\\1693+r+5&=1693+R\\r+5&=R\end{aligned}$$

Let's examine the median for different values of r.

Case 1: If $r\leq 182$, then $R\leq 187$ (since $R=r+5$).

The median with the incorrect height is $\frac{187+188}{2}=187.5$. Since R and r are both less than 188, the median with the correct height will remain at 187.5. Therefore, this case is not a possible solution.

Case 2: If $r=183$, then $R=188$.

The median with the incorrect height is $\frac{187+188}{2}=187.5$. Since $R=188$, the median with the correct height is $\frac{188+188}{2}=188$, which is an increase of 0.5 in the median. Therefore, this case is one possible solution.

Case 3: If $r = 184$, then $R = 189$.

The median with the incorrect height is $\frac{187+188}{2} = 187.5$. Since $R = 189$, the median with the correct height is $\frac{188+189}{2} = 188.5$, which is an increase of 1 in the median. Therefore, this case is not a possible solution.

Case 4: If $185 \leq r \leq 187$, then $190 \leq R \leq 192$.

The median with the incorrect height is $\frac{187+188}{2} = 187.5$. Since $190 \leq R \leq 193$, the median with the correct height is $\frac{188+190}{2} = 189$, which is an increase of 1.5 in the median. Therefore, this case is not a possible solution.

Case 5: If $r = 188$, then $R = 193$.

The median with the incorrect height is $\frac{188+188}{2} = 188$. Since $R = 193$, the median with the correct height is $\frac{188+190}{2} = 189$, which is an increase of 1 in the median. Therefore, this case is not a possible solution.

Case 6: If $r = 189$, then $R = 194$.

The median with the incorrect height is $\frac{188+189}{2} = 188.5$. Since $R = 194$, the median with the correct height is $\frac{188+190}{2} = 189$, which is an increase of 0.5 in the median. Therefore, this case is one possible solution.

Case 7: If $r \geq 190$, then $R \geq 195$.

The median with the incorrect height is $\frac{188+190}{2} = 189$. Since both r and R are greater than or equal to 190, the median with the correct height will remain at $\frac{188+190}{2} = 189$. Therefore, this case is not a possible solution.

Therefore, the possible correct heights for Richard are 188 cm or 194 cm.

Problem of the Week

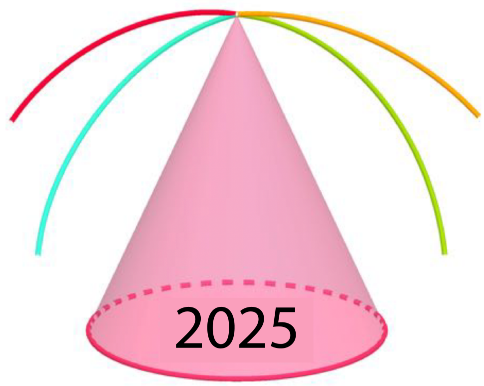
Problem D

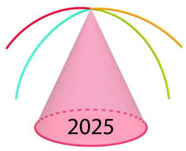
Welcome to a New Year!

5^3 is a *power* with *base* 5 and *exponent* 3.

5^3 means $5 \times 5 \times 5$ and is equal to 125 when expressed as an integer.

When $8^{674} \times 5^{2025}$ is expressed as an integer, how many digits are in the product?





Problem of the Week
Problem D and Solution
Welcome to a New Year!

Problem

5^3 is a *power* with *base* 5 and *exponent* 3.

5^3 means $5 \times 5 \times 5$ and is equal to 125 when expressed as an integer.

When $8^{674} \times 5^{2025}$ is expressed as an integer, how many digits are in the product?

Solution

An immediate temptation might be to reach for a calculator. In this case, basic calculator technology will let you down. We will solve this problem using our knowledge of powers and corresponding power laws.

$$\begin{aligned} 8^{674} \times 5^{2025} &= \left((2^3)^{674}\right) \times 5^{2025} \\ &= 2^{3\times 674} \times 5^{2025} \\ &= 2^{2022} \times 5^{2025} \\ &= 2^{2022} \times 5^{2022} \times 5^3 \\ &= (2 \times 5)^{2022} \times 125 \\ &= 10^{2022} \times 125 \end{aligned}$$

But 10^{2022} is the number 1 followed by 2022 zeroes. When we multiply this number by the three-digit number 125, we obtain the number 125 followed by 2022 zeroes. Therefore, $8^{674} \times 5^{2025}$ has $2022 + 3 = 2025$ digits. Happy New Year again!

Problem of the Week
Problem D
A Close Race

Joshua and Berhino are two Olympic long distance runners. Their qualifying times were equal ten weeks prior to the Olympics. Two weeks after qualifying, Joshua got an injury and his time increased by 3.2%, but he was able to decrease that time by 8.1% before the Olympics. Berhino's time decreased by 2% and then again by r% over the same time period.

If they finished the Olympic race with the same time, determine the value of r, rounded to the nearest tenth.

Problem of the Week
Problem D and Solution
A Close Race

Problem

Joshua and Berhino are two Olympic long distance runners. Their qualifying times were equal ten weeks prior to the Olympics. Two weeks after qualifying, Joshua got an injury and his time increased by 3.2%, but he was able to decrease that time by 8.1% before the Olympics. Berhino's time decreased by 2% and then again by r% over the same time period.

If they finished the Olympic race with the same time, determine the value of r, rounded to the nearest tenth.

Solution

Let t be Joshua's Olympic qualifying time. Since Joshua and Berhino had the same qualifying time, then t is also the qualifying time for Berhino.

Due to injury, Joshua's time increased by 3.2% two weeks after qualifying, so his time was

$$t + \frac{3.2}{100}t = \left(1 + \frac{3.2}{100}\right)t = 1.032t$$

He was able to decrease that time by 8.1% before the Olympics, so the new time is

$$1.032t - \left(\frac{8.1}{100}\right)(1.032t) = \left(1 - \frac{8.1}{100}\right)(1.032t) = 0.919(1.032t) = 0.948408t$$

Berhino's time first decreased by 2.0%, so his time was

$$t - \frac{2.0}{100}t = \left(1 - \frac{2.0}{100}\right)t = 0.98t$$

His time then further decreased by r%, so his time was

$$0.98t - \frac{r}{100}(0.98t) = 0.98t - \frac{0.98rt}{100}$$

Since they finished the Olympic race with the same time,

$$\begin{aligned} 0.948408t &= 0.98t - \frac{0.98rt}{100} \\ 0.948408t - 0.98t &= -\frac{0.98rt}{100} \\ 0.031592t &= \frac{0.98rt}{100} \end{aligned}$$

Dividing both sides by $t > 0$, we have

$$\begin{aligned} 0.031592 &= \frac{0.98r}{100} \\ 3.1592 &= 0.98r \\ r &\approx 3.2 \end{aligned}$$

Therefore, rounded to the nearest tenth, $r \approx 3.2$.

Problem of the Week

Problem D

It's Still Five

When an integer, N, is divided by 10, 11, or 12, the remainder is 5. If $N > 5$, what is the smallest possible value of N?

$$10\overline{)\,N} \qquad 11\overline{)\,N} \qquad 12\overline{)\,N}$$

Problem of the Week
Problem D and Solution
It's Still Five

Problem

When an integer, N, is divided by 10, 11, or 12, the remainder is 5. If $N > 5$, what is the smallest possible value of N?

$$10\overline{)N} \qquad 11\overline{)N} \qquad 12\overline{)N}$$

Solution

Solution 1

When N is divided by 10, 11, or 12, the remainder is 5. This means that $M = N - 5$ is divisible by each of 10, 11, and 12. Since M is divisible by each of 10, 11, and 12, then M is divisible by the least common multiple of 10, 11, and 12.

Since $10 = 2 \times 5$, $12 = 2^2 \times 3$, and 11 is prime, then to find the least common multiple, we calculate the product of the highest powers of each of the prime factors that occur in the given numbers. It follows that the least common multiple of 10, 11, and 12 is $2^2 \times 3 \times 5 \times 11 = 660$.

Since $N > 5$ and $M = N - 5$, we can conclude that $M > 0$. Since M is divisible by 660, then the smallest possible value for M is 660. Then $N = 660 + 5 = 665$.

Solution 2

When N is divided by 10, 11, or 12, the remainder is 5. This means that $M = N - 5$ is divisible by each of 10, 11, and 12. Since M is divisible by 10 and 11, then M must be divisible by the least common multiple of 10 and 11, which is 110. We test the first few multiples of 110 until we obtain one that is divisible by 12.

The integers 110, 220, 330, 440, and 550 are not divisible by 12, but 660 is. Therefore, M is divisible by 660. Since $N > 5$ and $M = N - 5$, we can conclude that $M > 0$. Therefore, the smallest possible value for M is 660. Then $N = 660 + 5 = 665$.

Problem of the Week

Problem D

Sticker Situation

Kendi has a large collection of vinyl stickers, and each sticker has an animal on it or an emoji on it (but not both). In this collection, 300 of the stickers have animals on them, and 1 out of 5 of all of the stickers have animals on them. She would like to add more stickers to her collection so that there are 3 stickers with animals on them out of every 10 stickers.

If she can buy the stickers in packages of 60 stickers where 21 are animal stickers and the remaining are emoji stickers, how many whole packages does she need to buy?

Problem of the Week

Problem D and Solution

Sticker Situation

Problem

Kendi has a large collection of vinyl stickers, and each sticker has an animal on it or an emoji on it (but not both). In this collection, 300 of the stickers have animals on them, and 1 out of 5 of all of the stickers have animals on them. She would like to add more stickers to her collection so that there are 3 stickers with animals on them out of every 10 stickers.

If she can buy the stickers in packages of 60 stickers where 21 are animal stickers and the remaining are emoji stickers, how many whole packages does she need to buy?

Solution

There were initially 300 animal stickers, and there was 1 animal sticker for every 5 stickers. This means that 4 out of 5 stickers were emoji stickers. Therefore, there were four times as many emoji stickers as animal stickers. That is, there were $4 \times 300 = 1200$ emoji stickers and a total of $300 + 1200 = 1500$ stickers.

Each package contains 21 animal stickers and 39 emoji stickers, for a total of $21 + 39 = 60$ stickers.

Let n represent the number of additional whole packages required to add to this collection so that there are 3 animal stickers out of every 10 of the stickers. By purchasing n packages, she is adding $60n$ stickers to her collection, of which $21n$ are animal stickers. Thus, she will have a total of $1500 + 60n$ stickers, of which $300 + 21n$ are animal stickers.

If 3 out of 10 of the stickers in her collection are animal stickers, then we have

$$\begin{aligned}
\frac{\text{the number of animal stickers}}{\text{the total number of stickers}} &= \frac{3}{10} \\
\frac{300 + 21n}{1500 + 60n} &= \frac{3}{10} \\
10(300 + 21n) &= 3(1500 + 60n) \\
3000 + 210n &= 4500 + 180n \\
30n &= 1500 \\
n &= 50
\end{aligned}$$

Therefore, 50 additional packages of stickers must be purchased so that 3 out of 10 of the stickers in her collection are animal stickers.

We can check this. After purchasing 50 additional packages of stickers, there would be $300 + 21(50) = 1350$ animal stickers and a total of $1500 + 60(50) = 4500$ stickers. Then, the ratio of animal stickers to the total number of stickers is $\frac{1350}{4500} = \frac{3}{10}$, as required.

Problem of the Week

Problem D

Birthday Cake Trouble

Harri is the manager of a pet store. For his birthday, his employees decided to surprise him with a birthday cake. The cake cost \$65.40 and everyone agreed to split the cost evenly. However, when it came time to collect the money, three of the employees were nowhere to be found. This meant that everyone else had to pay an additional \$1.09 to cover the cost of the cake. How many employees does Harri have at the pet store?

Problem of the Week
Problem D and Solution
Birthday Cake Trouble

Problem

Harri is the manager of a pet store. For his birthday, his employees decided to surprise him with a birthday cake. The cake cost \$65.40 and everyone agreed to split the cost evenly. However, when it came time to collect the money, three of the employees were nowhere to be found. This meant that everyone else had to pay an additional \$1.09 to cover the cost of the cake. How many employees does Harri have at the pet store?

Solution

Solution 1

Let n represent the number of employees that Harri has at the pet store. Then $(n-3)$ represents the number of employees who actually paid for the cake.

The n employees had each agreed to pay $\frac{\$65.40}{n}$. However, $(n-3)$ employees actually each paid $\frac{\$65.40}{n-3}$, which was more than the amount they had originally agreed to. The difference between the two amounts is \$1.09. It follows that

$$\begin{aligned}
\frac{65.40}{n-3} - \frac{65.40}{n} &= 1.09 \\
\frac{65.40}{n-3}(n)(n-3) - \frac{65.40}{n}(n)(n-3) &= 1.09(n)(n-3) \\
65.40n - 65.40(n-3) &= 1.09(n)(n-3) \\
65.40n - 65.40n + 196.2 &= 1.09(n)(n-3) \\
\frac{196.2}{1.09} &= n(n-3) \\
180 &= n(n-3)
\end{aligned}$$

From here we see that we are looking for two positive integers that differ by 3 and multiply to 180. We notice that $15 \times 12 = 180$, and $15 - 12 = 3$. In fact these are the only two positive integers that differ by 3 and multiply to 180. It follows that $n = 15$. Thus, Harri has 15 employees at the pet store.

Solution 2

This solution builds onto Solution 1 by solving the problem algebraically. Note that this level of mathematics is often not taught until grade 10.

Start with Solution 1 and proceed until you reach $180 = n(n-3)$. From there,

$$\begin{aligned}
180 &= n^2 - 3n \\
0 &= n^2 - 3n - 180 \\
0 &= (n-15)(n+12)
\end{aligned}$$

Therefore, $n = 15$ or $n = -12$. However, since $n > 0$, it follows that $n = 15$. Thus, Harri has 15 employees at the pet store.

Solution 3

Let n represent the number of employees that Harri has at the pet store. Then $(n-3)$ represents the number of employees who actually paid for the cake. The cost of the cake was \$65.40, or 6540 cents. Since 3 of the employees didn't pay, and at least one employee did pay, then we can assume there are at least 4 employees in total.

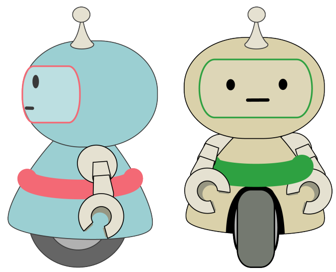
We check integer values of n, starting with $n = 4$, and determine the difference between the cost per person when there are n people compared to when there are $(n-3)$ people, until we find a difference of 109 cents. This is summarized in the table below.

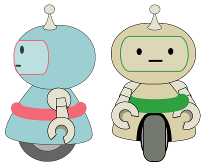
Number of Employees (n)	Original Amount, in cents, per Employee $\left(\frac{6540}{n}\right)$	Number of Employees who Paid $(n-3)$	Amount Actually Paid, in cents, per Employee $\left(\frac{6540}{n-3}\right)$	Difference in Amounts, in cents $\left(\frac{6540}{n-3} - \frac{6540}{n}\right)$
4	$\frac{6540}{4} = 1635$	$4-3=1$	$\frac{6540}{1} = 6540$	$6540 - 1635 = 4905$
5	$\frac{6540}{5} = 1308$	$5-3=2$	$\frac{6540}{2} = 3270$	$3270 - 1308 = 1962$
6	$\frac{6540}{6} = 1090$	$6-3=3$	$\frac{6540}{3} = 2180$	$2180 - 1090 = 1090$
7	$\frac{6540}{7} \approx 934.29$	$7-3=4$	$\frac{6540}{4} = 1635$	$1635 - 934.29 = 700.71$
8	$\frac{6540}{8} = 817.5$	$8-3=5$	$\frac{6540}{5} = 1308$	$1308 - 817.5 = 490.5$
9	$\frac{6540}{9} \approx 726.67$	$9-3=6$	$\frac{6540}{6} = 1090$	$1090 - 726.67 = 363.33$
10	$\frac{6540}{10} = 654$	$10-3=7$	$\frac{6540}{7} \approx 934.29$	$934.29 - 654 = 280.29$
11	$\frac{6540}{11} \approx 594.55$	$11-3=8$	$\frac{6540}{8} = 817.5$	$817.5 - 594.55 = 222.95$
12	$\frac{6540}{12} = 545$	$12-3=9$	$\frac{6540}{9} \approx 726.67$	$726.67 - 545 = 181.67$
13	$\frac{6540}{13} \approx 503.08$	$13-3=10$	$\frac{6540}{10} = 654$	$654 - 503.08 = 150.92$
14	$\frac{6540}{14} \approx 467.14$	$14-3=11$	$\frac{6540}{11} \approx 594.55$	$594.55 - 467.14 = 127.14$
15	$\frac{6540}{15} = 436$	$15-3=12$	$\frac{6540}{12} = 545$	$545 - 436 = 109$

Therefore, when $n = 15$, the difference in amounts is 109 cents, or \$1.09, as desired. Since the difference is decreasing as n is increasing, this is the only possible value of n. Thus, Harri has 15 employees at the pet store.

Problem of the Week
Problem D
Too Far

Berenice is testing two robots that she built. She programs one robot to travel north at 24 km/h and the other to travel east at 18 km/hr. She programs the robots so that once they are 15 km apart, they will stop moving. At exactly 1 p.m., she starts both robots from the same location. If the robots work as intended, at what time will they stop moving?





Problem of the Week
Problem D and Solution
Too Far

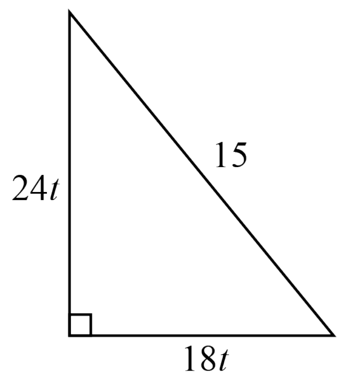
Problem

Berenice is testing two robots that she built. She programs one robot to travel north at 24 km/h and the other to travel east at 18 km/hr. She programs the robots so that once they are 15 km apart, they will stop moving. At exactly 1 p.m., she starts both robots from the same location. If the robots work as intended, at what time will they stop moving?

Solution

Let t be the time, in hours, that the two robots travel until they are 15 km apart. Since one robot is traveling at 24 km/h, it will travel $24t$ km in t hours. Since the other robot is traveling at 18 km/h, it will travel $18t$ km in t hours.

Since one robot is traveling north and the other is traveling east, they are traveling at right angles to each other. We can represent the distances traveled in kilometres on a right-angled triangle as shown.



Using the Pythagorean Theorem,

$$\begin{aligned}(24t)^2 + (18t)^2 &= 15^2 \\ 576t^2 + 324t^2 &= 225 \\ 900t^2 &= 225 \\ t^2 &= \frac{225}{900} = \frac{1}{4}\end{aligned}$$

Thus, $t = \sqrt{\frac{1}{4}} = \frac{1}{2}$, since $t > 0$. Therefore, after half an hour, or at 1:30 p.m., the two robots should be exactly 15 km apart and should stop moving.

Problem of the Week

Problem D

1225 is SUMthing Special

Did you know that 1225 can be written as the sum of ten consecutive integers?

That is,

$$1225 = 118 + 119 + 120 + 121 + 122 + 123 + 124 + 125 + 126 + 127$$

The notation below illustrates a mathematical short form used for writing the above sum. This notation is called *Sigma Notation*.

$$\sum_{i=118}^{127} i = 1225$$

How many ways can the number 1225 be expressed as the sum of an **odd** number of consecutive positive integers?

$$\sum_{i=118}^{127} i = 1225$$

Problem of the Week
Problem D and Solution
1225 is SUMthing Special

Problem
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How many ways can the number 1225 be expressed as the sum of an **odd** number of consecutive positive integers?

Solution

We will use the following idea to solve this problem: If there exists an odd number, k, of consecutive integers that sum to 1225, then k is a divisor of 1225.

Furthermore, if $kn = 1225$, then n is the mean (average) of the k integers, and will appear in the middle of the sequence of numbers summing to 1225.

Why is this true? Let's first consider $k = 5$.

Five consecutive integers can be expressed as $n-2$, $n-1$, n, $n+1$, and $n+2$, where n is an integer.
Their sum is $(n-2)+(n-1)+n+(n+1)+(n+2) = 5n$.
Therefore, $5n = 1225$ and $n = 245$. Thus, the middle term in the sum is 245, and the series is $243 + 244 + 245 + 246 + 247 = 1225$.

In general, if there are k consecutive integers, where k is odd, and n is the middle number in the sum, then there are $\frac{k-1}{2}$ integers less than n in the sum and $\frac{k-1}{2}$ integers greater than n in the sum. Furthermore, the first integer in the sum is $n-\frac{k-1}{2}$, the last integer in the sum is $n+\frac{k-1}{2}$, and we can write the sum of these integers in this way:

$$\left(n-\frac{k-1}{2}\right)+\cdots+(n-3)+(n-2)+(n-1)+n+(n+1)+(n+2)+(n+3)+\cdots+\left(n+\frac{k-1}{2}\right)$$

This simplifies to kn. Thus, if this sum is equal to 1225, then $kn = 1225$ and so k is an odd divisor of 1225.

Since $1225 = 5^2 7^2$, the positive divisors of 1225 are 1, 5, 7, 25, 35, 49, 175, 245 and 1225, which are all odd.

For each odd divisor, k, of 1225, we determine $n = \frac{1225}{k}$, which will be the middle term in the sum. The k integers that sum to 1225 will then be $\left(n-\frac{k-1}{2}\right)+\cdots+n+\cdots+\left(n+\frac{k-1}{2}\right)$. This is summarized in the table below.

Number of Integers (k)	Middle Integer (n)	Sum of Integers
1	1225	1225
5	245	$243+244+245+246+247$
7	175	$172+173+174+175+176+177+178$
25	49	$37+38+\cdots+49+\cdots+60+61$
35	35	$18+19+\cdots+35+\cdots+51+52$
49	25	$1+2+\cdots+25+\cdots+48+49$
175	7	$(-80)+(-70)+\cdots+7+\cdots+93+94$
245	5	$(-117)+(-116)+\cdots+5+\cdots+126+127$
1225	1	$(-611)+(-610)+\cdots+1+\cdots+612+613$

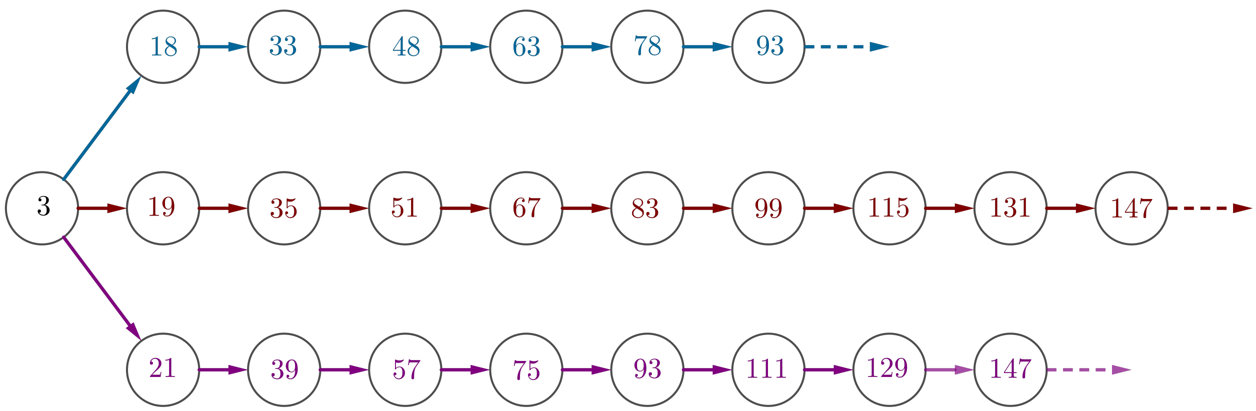
Note that all integers in the sum are positive for $k = 1, 5, 7, 25, 35, 49$. For $k = 175, 245, 1225$, there are negative integers in the sum.

Thus, there are six ways to express 1225 as the sum of an odd number of consecutive positive integers.

Extension: Determine the number of ways the number 1225 can be expressed as the sum of an **even** number of consecutive positive integers.

Problem of the Week
Problem D
Three Sequences

Three sequences each have first term 3. In the first sequence, each term after the first term is 15 more than the previous term. In the second sequence, each term after the first term is 16 more than the previous term. In the third sequence, each term after the first term is 18 more than the previous term. The three sequences continue indefinitely.



Determine all numbers between 3 and 2025, inclusive, that are common to all three sequences.

Problem of the Week

Problem D and Solution

Three Sequences

Problem

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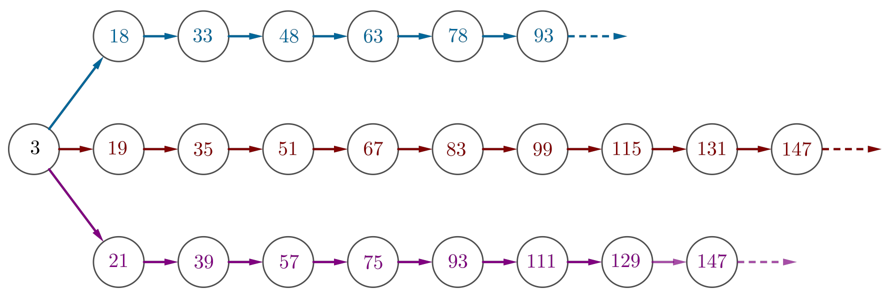


Determine all numbers between 3 and 2025, inclusive, that are common to all three sequences.

Solution

Solution 1

One could write out each sequence to a term less than or equal to 2025. You would write 135 terms of the first sequence, 127 terms of the second sequence and 113 terms of the third sequence. At this point you would compare the three sequences to find three numbers, 3, 723, and 1443, that are common to all three sequences. This is not a practical solution if you are solving the problem using pencil and paper. However, a solver could write a computer program that would easily handle this problem.

Solution 2

Notice that the number 147 occurs in both the second and third sequences. This number is $147 - 3 = 144$ greater than the first number common to both sequences. What is the significance of 144? It is the Least Common Multiple (LCM) of 16 and 18. The number 16 written in terms of prime factors is $2 \times 2 \times 2 \times 2 = 2^4$ and the number 18 written in terms of prime factors is $2 \times 3 \times 3 = 2 \times 3^2$. To find the LCM of 16 and 18, we determine the highest power that appears on each prime number in the two prime factorizations, and multiply all primes to the highest power together. Since the highest power of 2 in the two factorizations is 2^4 and the highest power of 3 in the two factorizations is 3^2, we have that the LCM of 16 and 18 is $2^4 \times 3^2 = 144$.

Now, we can create a fourth sequence that starts with 3, and each term after the first term is 144 greater than the previous term. The first few terms of this sequence are 3, 147, 291, 435, 579, 723, 867.

What numbers are common to this fourth sequence and the first sequence? We need to find the LCM of 15, the amount that each term in the first sequence increases by, and 144, the amount that each term in the fourth sequence increases by. The number 15 written in terms of prime factors is 3×5 and the number 144 written in terms of prime factors is $2^4 \times 3^2$. Therefore, the LCM of 15 and 144 is $2^4 \times 3^2 \times 5 = 720$.

Now, we create a fifth sequence that starts with 3, and each term after the first term is 720 greater than the previous term. This sequence contains all numbers that would be common to each of the three given sequences. The first few terms would be 3, 723, 1443, 2163.

Therefore, there are three numbers between 3 and 2025, inclusive, common to all three sequences. These numbers are 3, 723, and 1443.

Solution 3

This solution builds on the ideas in Solution 2. To solve the problem, we need to find the Least Common Multiple, or LCM, of 15, 16, and 18. First, we write each of the three numbers as a product of their prime factors.

$$
\begin{aligned}
15 &= 3 \times 5 \\
16 &= 2 \times 2 \times 2 \times 2 = 2^4 \\
18 &= 2 \times 3 \times 3 = 2 \times 3^2
\end{aligned}
$$

The LCM of 15, 16 and 18 is equal to the product of the highest power that appears on each prime number in the three prime factorizations. Since the highest power of 2 in the three factorizations is 2^4, the highest power of 3 is 3^2, and the highest power of 5 is 5^1, we have that the LCM of 15, 16, and 18 is $2^4 \times 3^2 \times 5 = 720$.

We can now determine the numbers between 3 and 2025 that would be common to all three sequences. The numbers are 3, $3 + 720 = 723$ and $723 + 720 = 1443$. If we were to continue, the next common number would be $1443 + 720 = 2163$, which is greater than 2025.

Therefore, there are three numbers between 3 and 2025, inclusive, common to all three sequences. These numbers are 3, 723, and 1443.

Problem of the Week

Problem D

Again in Reverse

A *palindrome* is a word or phrase which reads the same forwards and backwards, ignoring spaces and punctuation. Numbers which remain the same when the digits are reversed are also considered to be palindromes. For example 9 357 539, 6116, and 2 are all palindromes.

How many positive integers less than 1 000 000 are palindromes?

Problem of the Week

Problem D and Solution

Again in Reverse

Problem

A *palindrome* is a word or phrase which reads the same forwards and backwards, ignoring spaces and punctuation. Numbers which remain the same when the digits are reversed are also considered to be palindromes. For example 9 357 539, 6116, and 2 are all palindromes.

How many positive integers less than 1 000 000 are palindromes?

Solution

We consider cases, based on the number of digits in the palindrome. Since we are looking for positive integers less than 1 000 000, the minimum number of digits is one and the maximum is six.

- **Case 1:** One-digit palindromes
  Each of the integers from 1 to 9 is a palindrome. Thus, there are 9 one-digit palindromes.

- **Case 2:** Two-digit palindromes
  In order to be a palindrome, the two digits must be the same. Therefore, the integer must be of the form aa, where a is an integer from 1 to 9. Thus, there are 9 two-digit palindromes.

- **Case 3:** Three-digit palindromes
  In order to be a palindrome, the first and last digits must be the same. Therefore, the integer must be of the form aba, where a is an integer from 1 to 9 and b is an integer from 0 to 9. There are 9 choices for a, and for each of these choices there are 10 choices for b. Thus, there are $9 \times 10 = 90$ three-digit palindromes.

- **Case 4:** Four-digit palindromes
  In order to be a palindrome, the first and last digits must be the same, and the second and third digits must be the same. Therefore, the integer must be of the form $abba$, where a is an integer from 1 to 9 and b is an integer from 0 to 9. As with the previous case, there are 9 choices for a, and for each of these choices there are 10 choices for b. Thus, there are $9 \times 10 = 90$ four-digit palindromes.

- **Case 4:** Five-digit palindromes
  In order to be a palindrome, the first and last digits must be the same, and the second and fourth digits must be the same. Therefore, the integer must be of the form $abcba$, where a is an integer from 1 to 9 and b and c are integers from 0 to 9. There are 9 choices for a, for each of these choices there are 10 choices for b, and for each of these choices there are 10 choices for c. Thus, there are $9 \times 10 \times 10 = 900$ five-digit palindromes.

- **Case 4:** Six-digit palindromes
  In order to be a palindrome, the first and last digits must be the same, the second and fifth digits must be the same, and the third and fourth digits must be the same. Therefore, the integer must be of the form $abccba$, where a is an integer from 1 to 9 and b and c are integers from 0 to 9. As with the previous case, there are 9 choices for a, for each of these choices there are 10 choices for b, and for each of these choices there are 10 choices for c. Thus, there are $9 \times 10 \times 10 = 900$ six-digit palindromes.

Thus, there are $9 + 9 + 90 + 90 + 900 + 900 = 1998$ palindromes less than $1\,000\,000$.

Problem of the Week

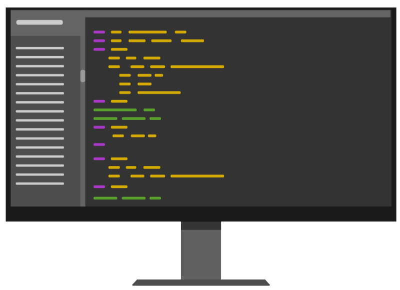
Problem D

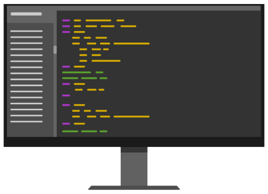
Another Program

To practice her programming skills, Tessa wrote a program that takes an input number, squares it, adds a constant value, multiplies the result by another constant value, and then outputs the result.

When the input is 8, the output is 204. When the input is 3, the output is 39.

What is the output when the input is 5?





Problem of the Week
Problem D and Solution
Another Program

Problem

To practice her programming skills, Tessa wrote a program that takes an input number, squares it, adds a constant value, multiplies the result by another constant value, and then outputs the result.

When the input is 8, the output is 204. When the input is 3, the output is 39.

What is the output when the input is 5?

Solution

Let the first constant value be k and the second constant value be p. Then Tessa's program takes an input number, squares it, adds k, multiplies the result by p, and then outputs the result.

Since an input of 8 gives an output of 204,

$$\begin{aligned}(8^2 + k) \times p &= 204 \\ p &= \frac{204}{64 + k} \qquad (1)\end{aligned}$$

Similarly, since an input of 3 gives an output of 39,

$$\begin{aligned}(3^2 + k) \times p &= 39 \\ p &= \frac{39}{9 + k} \qquad (2)\end{aligned}$$

From equations (1) and (2), we can conclude the following.

$$\begin{aligned}\frac{204}{64 + k} &= \frac{39}{9 + k} \\ 204(9 + k) &= 39(64 + k) \\ 1836 + 204k &= 2496 + 39k \\ 165k &= 660 \\ k &= \frac{660}{165} = 4\end{aligned}$$

Then $p = \dfrac{39}{9 + k} = \dfrac{39}{9 + 4} = 3$.

Now that we have determined the values of k and p, we can determine the output when the input is 5.

$$(5^2 + 4) \times 3 = 29 \times 3 = 87$$

Thus, the output is 87 when the input is 5.