$$\sum_{i=118}^{127} i = 1225$$

Problem of the Week
Problem D and Solution
1225 is SUMthing Special

Problem
Did you know that 1225 can be written as the sum of ten consecutive integers?

That is,
$$1225 = 118 + 119 + 120 + 121 + 122 + 123 + 124 + 125 + 126 + 127$$

How many ways can the number 1225 be expressed as the sum of an **odd** number of consecutive positive integers?

Solution

We will use the following idea to solve this problem: If there exists an odd number, k, of consecutive integers that sum to 1225, then k is a divisor of 1225.

Furthermore, if $kn = 1225$, then n is the mean (average) of the k integers, and will appear in the middle of the sequence of numbers summing to 1225.

Why is this true? Let's first consider $k = 5$.

Five consecutive integers can be expressed as $n - 2$, $n - 1$, n, $n + 1$, and $n + 2$, where n is an integer.
Their sum is $(n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 5n$.
Therefore, $5n = 1225$ and $n = 245$. Thus, the middle term in the sum is 245, and the series is $243 + 244 + 245 + 246 + 247 = 1225$.

In general, if there are k consecutive integers, where k is odd, and n is the middle number in the sum, then there are $\frac{k-1}{2}$ integers less than n in the sum and $\frac{k-1}{2}$ integers greater than n in the sum. Furthermore, the first integer in the sum is $n - \frac{k-1}{2}$, the last integer in the sum is $n + \frac{k-1}{2}$, and we can write the sum of these integers in this way:

$$\left(n - \frac{k-1}{2}\right) + \cdots + (n-3) + (n-2) + (n-1) + n + (n+1) + (n+2) + (n+3) + \cdots + \left(n + \frac{k-1}{2}\right)$$

This simplifies to kn. Thus, if this sum is equal to 1225, then $kn = 1225$ and so k is an odd divisor of 1225.

Since $1225 = 5^2 7^2$, the positive divisors of 1225 are 1, 5, 7, 25, 35, 49, 175, 245 and 1225, which are all odd.

For each odd divisor, k, of 1225, we determine $n = \frac{1225}{k}$, which will be the middle term in the sum. The k integers that sum to 1225 will then be $\left(n - \frac{k-1}{2}\right) + \cdots + n + \cdots + \left(n + \frac{k-1}{2}\right)$. This is summarized in the table below.

Number of Integers (k)	Middle Integer (n)	Sum of Integers
1	1225	1225
5	245	$243 + 244 + 245 + 246 + 247$
7	175	$172 + 173 + 174 + 175 + 176 + 177 + 178$
25	49	$37 + 38 + \cdots + 49 + \cdots + 60 + 61$
35	35	$18 + 19 + \cdots + 35 + \cdots + 51 + 52$
49	25	$1 + 2 + \cdots + 25 + \cdots + 48 + 49$
175	7	$(-80) + (-70) + \cdots + 7 + \cdots + 93 + 94$
245	5	$(-117) + (-116) + \cdots + 5 + \cdots + 126 + 127$
1225	1	$(-611) + (-610) + \cdots + 1 + \cdots + 612 + 613$

Note that all integers in the sum are positive for $k = 1, 5, 7, 25, 35, 49$. For $k = 175, 245, 1225$, there are negative integers in the sum.

Thus, there are six ways to express 1225 as the sum of an odd number of consecutive positive integers.

Extension: Determine the number of ways the number 1225 can be expressed as the sum of an **even** number of consecutive positive integers.