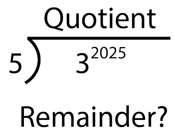

Problem of the Week
Problem C and Solution
What is the Remainder?

Problem

3^6 means $3 \times 3 \times 3 \times 3 \times 3 \times 3$ and is equal to 729 when expressed as an integer. When 3^6 is divided by 5, the remainder is 4.

When the integer 3^{2025} is divided by 5, what is the remainder?

Solution

When the units digit of an integer is 0 or 5, the remainder is 0 when divided by 5. When the units digit of an integer is 1 or 6, the remainder is 1 when divided by 5. For example, $56 \div 5 = 11$ R1. When the units digit of an integer is 2 or 7, the remainder is 2 when divided by 5. When the units digit of an integer is 3 or 8, the remainder is 3 when divided by 5. When the units digit of an integer is 4 or 9, the remainder is 4 when divided by 5.

Let's examine the pattern of the units digits of the first eight powers of 3.

Power	Value	Units Digit
3^1	3	3
3^2	9	9
3^3	27	7
3^4	81	1
3^5	243	3
3^6	729	9
3^7	2187	7
3^8	6561	1

The units digits appears to repeat every four powers of 3. To convince ourselves that this pattern continues, we notice that when starting with the integer 3 and continually multiplying by 3, there are a limited number of options for the resulting product's units digit. For any integer whose units digit is 3, when multiplied by 3, the resulting product's units digit is 9. For any integer whose units digit is 9, when multiplied by 3, the resulting product's units digit is 7. For any integer whose units digit is 7, when it is multiplied by 3, the resulting product's units digit is 1. For any integer whose units digit is 1, when it is multiplied by 3, the resulting product's units digit is 3. Thus, the cycle repeats.

Another way to convince ourselves that the pattern continues is by considering the fact that any positive integer can be written as $10a + b$, where a and b are integers and $a \geq 0$ and $0 \leq b \leq 9$. Its product with 3 can be written as $3(10a + b) = 3(10a) + 3b$. Since $3(10a) = 3a(10)$, it must have a units digit of 0. Therefore, the units digit of $3(10a + b)$ must be the same as the units digit of $3b$. Thus, for any integer with units digit is 3 (so $b = 3$), when multiplied by 3, the resulting product's units digit is 9. For any integer whose units digit is 9 (so $b = 9$), when multiplied by 3, the resulting product's units digit is 7. For any integer whose units digit is 7 (so $b = 7$), when multiplied by 3, the resulting product's units digit is 1. For any integer whose units digit is 1 (so $b = 1$), when multiplied by 3, the resulting product's units digit is 3. Since the first power of 3 in our list, 3^1, has $b = 3$, we get the repeating cycle 3, 9, 7, and 1 of units digits.

Thus, we need to determine how many cycles of four there are in 2025 and if any remainder exists. We determine that $2025 \div 4 = 506$ R1. The remainder 1 corresponds to the first term of the cycle whose units digit is 3, meaning that the units digit of 3^{2025} is 3. Since the units digit is 3, the remainder when 3^{2025} is divided by 5 is 3.

Therefore, when 3^{2025} is divided by 5, the remainder is 3.