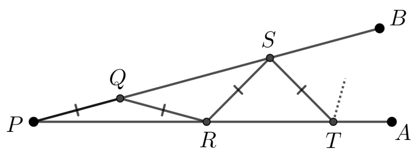


Problem of the Week
Problem C and Solution
Back and Forth

Problem

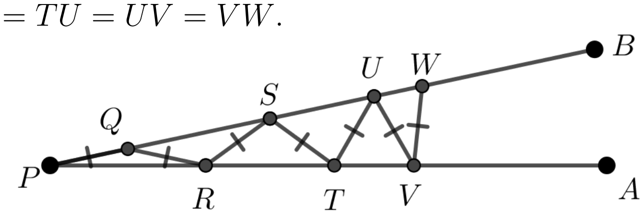
Line segments BP and AP are such that $\angle BPA = 12^\circ$. Points Q, R, S, T, . . . alternate from one arm of the angle to the other, with Q on BP and R on AP, such that each point is located farther away from P than the point before, and $PQ = QR = RS = ST = \ldots$. This creates isosceles triangles $\triangle PQR$, $\triangle QRS$, $\triangle RST$, and so on. Eventually, one of the isosceles triangles will also be an equilateral triangle.

How many isosceles triangles will be created before the equilateral triangle is created?

Solution

We will show that there will be 4 isosceles triangles created before the equilateral triangle is created. Throughout this solution there will be references to the Exterior Angle Theorem, which states: "The measure of an exterior angle of a triangle is equal to the sum of the measures of the opposite interior angles." For example, $\angle SQR$ is exterior to $\triangle QPR$ and $\angle SQR = \angle QPR + \angle QRP$.

We extend the diagram to include U on BP, V on AP, and W on BP such that $PQ = QR = RS = ST = TU = UV = VW$.



In $\triangle QPR$, $\angle QPR = 12^\circ$ and $PQ = QR$. Therefore, $\triangle QPR$ is isosceles and $\angle QRP = \angle QPR = 12^\circ$. Since $\angle SQR$ is exterior to $\triangle QPR$, by the Exterior Angle Theorem, $\angle SQR = \angle QPR + \angle QRP = 12^\circ + 12^\circ = 24^\circ$.

In $\triangle RQS$, $\angle SQR = 24^\circ$ and $QR = RS$. Therefore, $\triangle RQS$ is isosceles and $\angle RSQ = \angle SQR = 24^\circ$. Since $\angle SRT$ is exterior to $\triangle PRS$, by the Exterior Angle Theorem, $\angle SRT = \angle SPR + \angle PSR = 12^\circ + 24^\circ = 36^\circ$.

In $\triangle SRT$, $\angle SRT = 36^\circ$ and $SR = ST$. Therefore, $\triangle SRT$ is isosceles and $\angle STR = \angle SRT = 36^\circ$. Since $\angle UST$ is exterior to $\triangle PST$, by the Exterior Angle Theorem, $\angle UST = \angle SPT + \angle STP = 12^\circ + 36^\circ = 48^\circ$.

In $\triangle TSU$, $\angle UST = 48^\circ$ and $ST = TU$. Therefore, $\triangle TSU$ is isosceles and $\angle TUS = \angle UST = 48^\circ$. Since $\angle UTV$ is exterior to $\triangle PUT$, by the Exterior Angle Theorem, $\angle UTV = \angle UPT + \angle PUT = 12^\circ + 48^\circ = 60^\circ$.

In $\triangle UTV$, $\angle UTV = 60^\circ$ and $TU = UV$. Therefore, $\triangle UTV$ is isosceles and $\angle UVT = \angle UTV = 60^\circ$. Since $\angle UVT = \angle UTV = 60^\circ$, the remaining angle in $\triangle UTV$ is $\angle TUV = 180^\circ - 60^\circ - 60^\circ = 60^\circ$. Since all three angles equal 60°, $\triangle UTV$ is equilateral.

Since $\triangle UTV$ is the fifth isosceles triangle created, there are 4 isosceles triangles created before the equilateral triangle is created.

Extension: In our problem, we started with $\angle APB = 12^\circ$ and we eventually created an equilateral triangle. What other values for $\angle APB$ will eventually create an equilateral triangle?