Problem of the Week

Problems and Solutions 2024-2025

Problem C

Grade 7/8

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

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Algebra (A)

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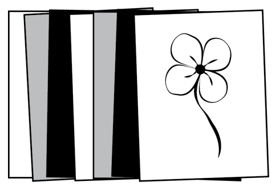
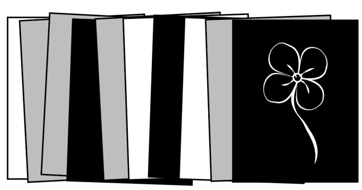
Problem C
Selling Sketches

Elouise sketches pictures and sells them in the park during a summer festival. She uses three colours of paper: white, grey, and black. First she sketches on white paper and puts it on the table for display. Then she sketches on grey paper, and then she sketches on black paper. After she has finished a sketch, she places it on top of the display pile. She continues in this pattern of white, grey, black, so that after completing seven sketches, her display pile would be as shown.



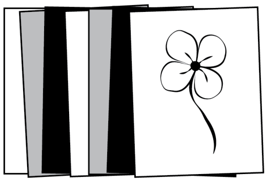
Elouise starts the festival with nothing in her display pile and continues sketching and adding pictures to the display pile throughout the festival. If someone wants to buy one of her sketches, they can only buy the sketch that is currently on the top of the pile. At the end of the festival, Elouise notes that she has sketched more pictures than she has sold, and the remaining sketches in her display pile are as shown.



What is the fewest number of sketches that Elouise could have sold during the festival?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

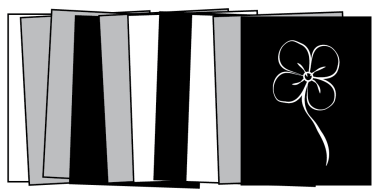
Themes: Algebra, Computational Thinking



Problem of the Week
Problem C and Solution
Selling Sketches

Problem

Elouise sketches pictures and sells them in the park during a summer festival. She uses three colours of paper: white, grey, and black. First she sketches on white paper and puts it on the table for display. Then she sketches on grey paper, and then she sketches on black paper. After she has finished a sketch, she places it on top of the display pile. She continues in this pattern of white, grey, black, so that after completing seven sketches, her display pile would be as shown above. Elouise starts the festival with nothing in her display pile and continues sketching and adding pictures to the display pile throughout the festival. If someone wants to buy one of her sketches, they can only buy the sketch that is currently on the top of the pile. At the end of the festival, Elouise notes that she has sketched more pictures than she has sold, and the remaining sketches in her display pile are as shown.



What is the fewest number of sketches that Elouise could have sold during the festival?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Solution

Let W represent a sketch on white paper, G represent a sketch on grey paper, and B represent a sketch on black paper. Then the sequence in the display pile at the end of the festival is $WGGBGWBWGB$, where the leftmost sketches represent those on the bottom of the pile.

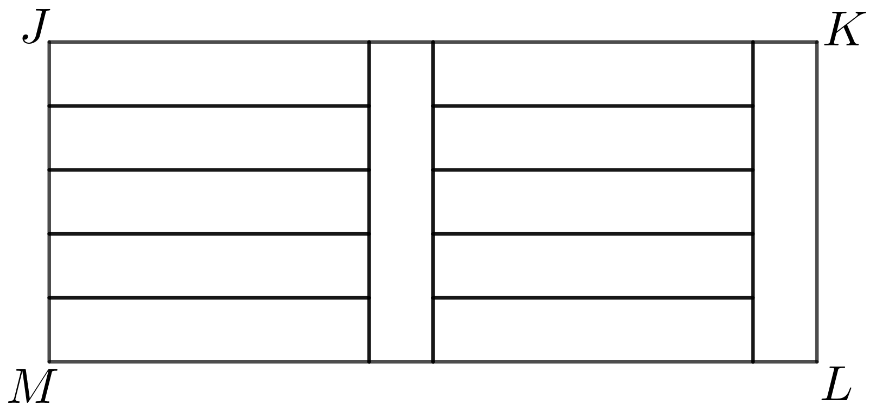
If no sketches were sold at all, the sequence would be $WGBWGBWGB\ldots$.

We will place a _ in our display pile sequence at the places where we know some sketches must have been sold, because it does not follow the sequence in which they were made. This gives $WG\_GB\_G\_W\_BWGB$. Moving from left to right, the fewest number of sketches that could fill the first blank is 2, namely BW. The fewest number of sketches that could fill the second blank is 1, namely W. The fewest number of sketches that could fill the third blank is 1, namely B. The fewest number of sketches that could fill the last blank is 1, namely G. Filling these in gives $WG\underline{BW}GB\underline{W}G\underline{B}W\underline{G}BWGB$, as desired.

Therefore, the minimum number of sketches that Elouise could have sold is $2 + 1 + 1 + 1 = 5$.

Problem of the Week
Problem C
A Rectangle of Rectangles

Large rectangle $JKLM$ is formed by twelve identical smaller rectangles, as shown.



If the area of $JKLM$ is 540 cm^2, then determine the dimensions of the smaller rectangles.

Themes: Algebra, Geometry & Measurement

Problem of the Week
Problem C and Solution
A Rectangle of Rectangles

Problem
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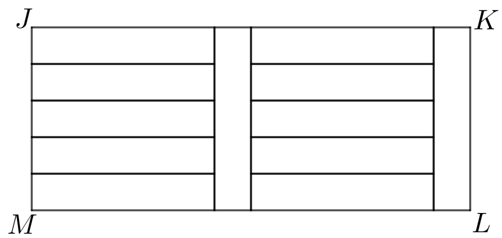
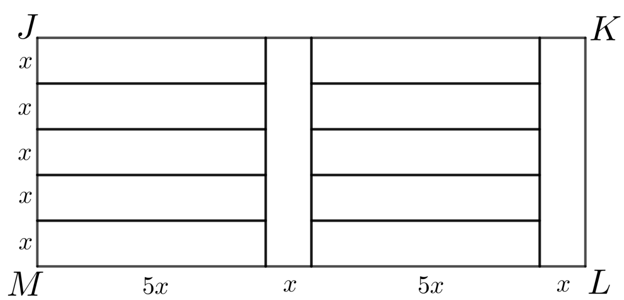


If the area of $JKLM$ is 540 cm^2, then determine the dimensions of the smaller rectangles.

Solution
Let x be the width of one of the smaller identical rectangles, in cm. Five of the smaller rectangles are stacked on top of each other forming JM, so $JM = x + x + x + x + x = 5x$. Since $JKLM$ is a rectangle, $JM = KL = 5x$. Thus, $5x$ is also the length of a smaller rectangle. Therefore, a smaller rectangle is $5x$ cm by x cm.



From here, we proceed with two different solutions.

Solution 1
Since $JKLM$ is formed by twelve identical smaller rectangles, the area of rectangle $JKLM$ is equal to 12 times the area of one of the smaller rectangles.

$$\begin{aligned} \text{Area } JKLM &= 12 \times \text{Area of one smaller rectangle} \\ 540 &= 12 \times 5x \times x \\ 540 &= 60 \times x^2 \end{aligned}$$

Dividing both sides by 60, we obtain $x^2 = 9$. Since x is the width of a smaller rectangle, $x > 0$, and so $x = 3$ follows.

Thus, the width of a smaller rectangle is $x = 3$ cm and the length of a smaller rectangle is $5x = 5(3) = 15$ cm.

Therefore, the smaller rectangles are each 15 cm by 3 cm.

Solution 2

Side length ML is made up of the lengths of two of the smaller rectangles plus the widths of two of the smaller rectangles. Therefore, $ML = 5x + 5x + x + x = 12x$ and rectangle $JKLM$ is $12x$ cm by $5x$ cm.

To find the area of $JKLM$ we multiply the length ML by the width JM.

$$\begin{aligned} \text{Area } JKLM &= ML \times JM \\ 540 &= (12x) \times (5x) \\ 540 &= 12 \times 5 \times x \times x \\ 540 &= 60 \times x^2 \end{aligned}$$

Dividing both sides by 60, we obtain $x^2 = 9$. Since x is the width of a smaller rectangle, $x > 0$, and so $x = 3$ follows.

Thus, the width of a smaller rectangle is $x = 3$ cm and the length of a smaller rectangle is $5x = 5(3) = 15$ cm.

Therefore, the smaller rectangles are each 15 cm by 3 cm.

Problem of the Week

Problem C

The Sequence of the Year

In a sequence of numbers, each number in the sequence is called a *term*. In the sequence 2, 4, 6, 8, the first term is 2, the second term is 4, the third term is 6, and the fourth term is 8.

In another sequence, the first term is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

What is the 2024$^{\text{th}}$ term in the sequence?

Themes: Algebra, Number Sense

Problem of the Week
Problem C and Solution
The Sequence of the Year

Problem

In a sequence of numbers, each number in the sequence is called a *term*. In the sequence 2, 4, 6, 8, the first term is 2, the second term is 4, the third term is 6, and the fourth term is 8.

In another sequence, the first term is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

What is the 2024^{th} term in the sequence?

Solution

We will begin by finding more terms in the sequence. The first 14 terms of the sequence are 24, 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1.

If we continue the sequence, we will see that the terms 4, 2, and 1 will continue to repeat. Thus, the 9^{th} term, 12^{th} term, 15^{th} term, and so on, will each have a value of 4. Notice that these term numbers are all multiples of 3. It follows that every term number after 9 that is a multiple of 3 will have a value of 4.

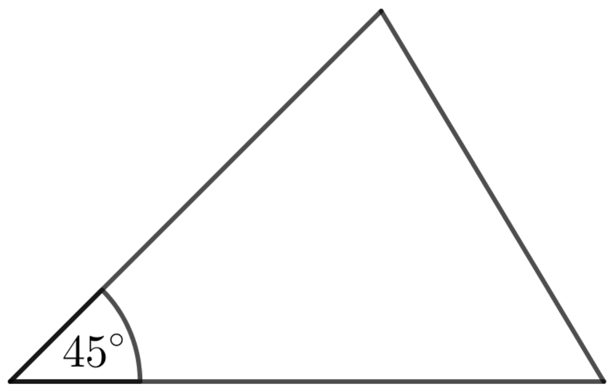
Thus, since 2022 is a multiple of 3, the 2022^{nd} term will have a value of 4. Then, the 2023^{rd} term will have a value of 2, and the 2024^{th} term will have a value of 1.

Extension:

In 1937, the mathematician Lothat Collatz wondered if any sequence whose terms after the first are determined in this way would always eventually reach the number 1, regardless of which number you started with. This problem is actually still unsolved today and is called the Collatz Conjecture.

Problem of the Week
Problem C
Two Out of Three Angles

While measuring the angles in a triangle, Patricia found the measure of one of the angles is 45°. Once she had measured the other two angles, she noticed that the measures of these two angles are in the ratio $4:5$. What is the measure of each of the other two angles?



Themes: Algebra, Geometry & Measurement, Number Sense

Problem of the Week
Problem C and Solution
Two Out of Three Angles

Problem

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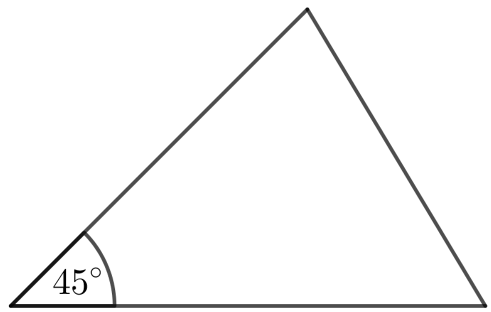


Solution

Solution 1

Since the measure of the unknown angles are in the ratio of $4:5$, then we can give the angles measures of $5n^\circ$ and $4n^\circ$.

Since the sum of the measures of the three angles in any triangle is 180°, then

$$\begin{aligned} 45 + 4n + 5n &= 180 \\ 4n + 5n &= 135 \\ 9n &= 135 \\ n &= 15 \end{aligned}$$

Therefore, the other two angles are $5n = 5(15) = 75^\circ$ and $4n = 4(15) = 60^\circ$.

Solution 2

Since the sum of the measures of the three angles in any triangle is 180°, then the sum of the measures of the two unknown angles in the triangle is $180^\circ - 45^\circ = 135^\circ$.

The measures of the two unknown angles are in the ratio $4:5$, and so one of the two angles measures $\frac{5}{4+5} = \frac{5}{9}$ of the sum of the two angles, while the other angle measures $\frac{4}{4+5} = \frac{4}{9}$ of the sum of the two angles.

That is, the larger of the two unknown angles measures $\frac{5}{9} \times 135^\circ = 75^\circ$, and the smaller of the unknown angles measures $\frac{4}{9} \times 135^\circ = 60^\circ$.

Therefore, the other two angles are 75° and 60°.

Problem of the Week

Problem C

Moving Time

Stefania's family is looking for a new apartment and Stefania is hoping to get a dog after they move. Of the 65 apartments she has found online, 40 are close to a park, 28 allow dogs, and 8 are not close to a park and do not allow dogs. How many of the apartments are close to a park and allow dogs?

Themes: Algebra, Data Management

Problem of the Week

Problem C and Solution

Moving Time

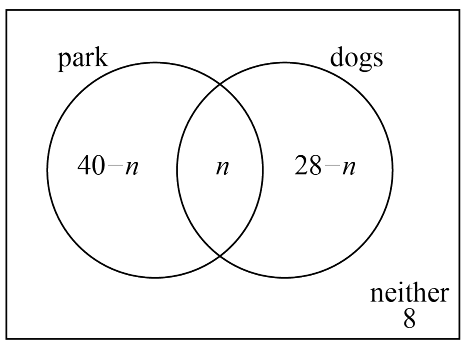
Problem

Stefania's family is looking for a new apartment and Stefania is hoping to get a dog after they move. Of the 65 apartments she has found online, 40 are close to a park, 28 allow dogs, and 8 are not close to a park and do not allow dogs. How many of the apartments are close to a park and allow dogs?

Solution

Solution 1

Let n be the number of apartments that are close to a park and allow dogs. Since 40 apartments are close to a park and n apartments are close to a park and allow dogs, then $40 - n$ apartments are close to a park but do not allow dogs. Similarly, since 28 apartments allow dogs and n apartments are close to a park and allow dogs, then $28 - n$ apartments allow dogs but are not close to a park. We also know that 8 apartments are not close to a park and do not allow dogs. This information is summarized in the following Venn diagram.



Since there are 65 apartments in total, then

$$\begin{aligned} 65 &= (40 - n) + n + (28 - n) + 8 \\ 65 &= 76 - n \\ n &= 76 - 65 = 11 \end{aligned}$$

Therefore, 11 of the apartments are close to a park and allow dogs.

Solution 2

Since 8 of the 65 apartments are not close to a park and do not allow dogs, we know that $65 - 8 = 57$ apartments must be close to a park or allow dogs, or both. We know that the 40 apartments close to a park will include the apartments that are close to a park and also allow dogs. Similarly, the 28 apartments that allow dogs will include the apartments that are close to a park and also allow dogs. Thus, if we consider $40 + 28 = 68$, we will be double counting the apartments that are close to a park and also allow dogs. Since there are 57 apartments that are close to a park or allow dogs, or both, we have double-counted $68 - 57 = 11$ apartments. Thus, there must be 11 apartments that are both close to a park and allow dogs.

Problem of the Week

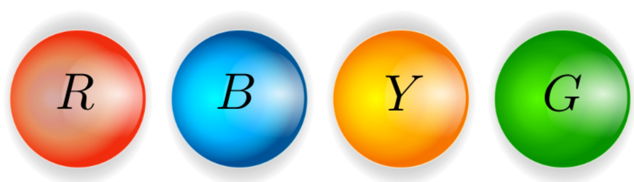
Problem C

Marbles, Marbles

A box contains 6 red marbles, 5 blue marbles, 2 yellow marbles, and 3 green marbles. Several orange marbles are added to the box. All the marbles in the box are identical except for colour.

A marble is then randomly selected from the box, and the probability that a blue or green marble is selected is $\frac{2}{7}$.

How many orange marbles were added to the box?



Themes: Algebra, Data Management

Problem of the Week
Problem C and Solution
Marbles, Marbles

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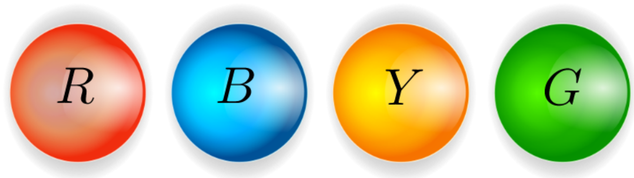

Solution

The number of blue and green marbles in the box is $5 + 3 = 8$.

Let n be the total number of marbles in the box after adding some orange marbles. Since the probability of picking a blue or green marble is $\frac{2}{7}$, we must have $\frac{8}{n} = \frac{2}{7}$.

If we multiply the numerator and denominator of the fraction $\frac{2}{7}$ by 4, we obtain $\frac{2}{7} = \frac{2\times4}{7\times4} = \frac{8}{28}$. Therefore, $\frac{8}{n} = \frac{8}{28}$. Since the fractions are equal and the numerators are equal, the denominators must also be equal. It follows that $n = 28$.

In the beginning, there were $5 + 6 + 3 + 2 = 16$ marbles in the box. Since there were 16 marbles in the box and there are now 28 marbles in the box, then $28 - 16 = 12$ orange marbles were added to the box.

Therefore, 12 orange marbles were added to the box.

Problem of the Week

Problem C

1000 is SUM Number!

Did you know that 1000 can be written as the sum of 16 consecutive integers?

That is,

$$1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70$$

The notation below illustrates a mathematical short form used for writing the above sum. The notation is called *Sigma Notation*.

$$\sum_{i=55}^{70} i = 1000$$

Using at least two integers, what is the minimum number of consecutive integers that sum to exactly 1000?

Themes: Algebra, Number Sense

$$\sum_{i=55}^{70} i = 1000$$

Problem of the Week
Problem C and Solution
1000 is SUM Number!

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Did you know that 1000 can be written as the sum of 16 consecutive integers?

That is,

$$1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70$$

Using at least two integers, what is the minimum number of consecutive integers that sum to exactly 1000?

Solution

We will start with using two integers and then increase the number of integers by 1, until we discover the first number of integers that works.

- Can 1000 be written as the sum of two consecutive integers?

  Let n and $n+1$ represent the two integers. Then we have

$$\begin{aligned} n + (n+1) &= 1000 \\ 2n + 1 &= 1000 \\ 2n &= 999 \\ n &= 499.5 \end{aligned}$$

  Since n is not an integer, it is not possible to write 1000 using two consecutive integers.

- Can 1000 be written as the sum of three consecutive integers?

  Let n, $n+1$, $n+2$ represent the three integers. Then we have

$$\begin{aligned} n + (n+1) + (n+2) &= 1000 \\ 3n + 3 &= 1000 \\ 3n &= 997 \\ n &\approx 332.3 \end{aligned}$$

  Since n is not integer, it is not possible to write 1000 using three consecutive integers. (Refer to the note following the solution for an alternate way to represent three consecutive integers.)

- Can 1000 be written as the sum of four consecutive integers?

  Let n, $n+1$, $n+2$, $n+3$ represent the four integers. Then we have

$$\begin{aligned} n + (n+1) + (n+2) + (n+3) &= 1000 \\ 4n + 6 &= 1000 \\ 4n &= 994 \\ n &= 248.5 \end{aligned}$$

  Since n is not an integer, it is not possible to write 1000 using four consecutive integers.

- Can 1000 be written as the sum of five consecutive integers?

  Let n, $n+1$, $n+2$, $n+3$, $n+4$ represent the five integers. Then we have

$$\begin{aligned} n+(n+1)+(n+2)+(n+3)+(n+4) &= 1000 \\ 5n+10 &= 1000 \\ 5n &= 990 \\ n &= 198 \end{aligned}$$

  Since n is an integer, it is possible to write 1000 using five consecutive integers. That is, as $1000 = 198 + 199 + 200 + 201 + 202$.

Therefore, the minimum number of consecutive integers that sum to 1000 is five.

Note:

In the second case, when we checked if 1000 could be written as the sum of three consecutive integers, we could have proceeded as follows:

Let $a-1$, a, $a+1$ represent the three consecutive integers. Then we have

$$\begin{aligned} (a-1)+a+(a+1) &= 1000 \\ 3a &= 1000 \\ a &\approx 333.3 \end{aligned}$$

Since a is not an integer, it is not possible to write 1000 using three consecutive integers.

This idea is useful when we are finding the sum of an odd number of consecutive integers. If we applied the same idea to the fourth case by using $a-2$, $a-1$, a, $a+1$, $a+2$ to represent the five consecutive integers, we would have

$$\begin{aligned} (a-2)+(a-1)+a+(a+1)+(a+2) &= 1000 \\ 5a &= 1000 \\ a &= 200 \end{aligned}$$

Thus, $a-2=198$, $a-1=199$, $a+1=201$, and $a+2=202$. And so it is possible to write 1000 using five consecutive integers as $1000 = 198 + 199 + 200 + 201 + 202$.

For Further Thought:

What is the largest **odd** number of consecutive positive integers that can be used to sum to 1000?

How would your answer change if the word positive was removed from the above sentence?

Problem of the Week

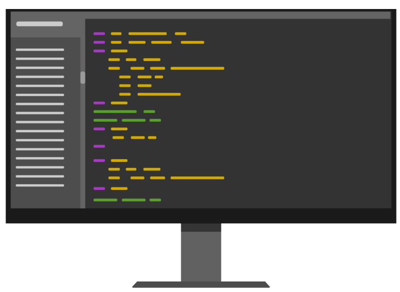
Problem C

Sum Program

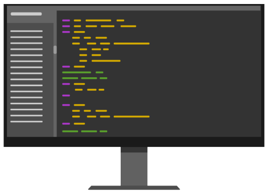
Ruben wrote a program that takes a list of numbers as input. The program then adds each pair of adjacent numbers in the list to obtain a new list of numbers, then repeats this process with the new list until the list contains only one number. This number is then the output.

For example, if the input is $(1,\ 5,\ 3,\ 2)$ then the program adds the adjacent numbers in the list to obtain $(6,\ 8,\ 5)$. The program then repeats this process to obtain $(14,\ 13)$, and then (27). Since the list now contains only one number, the output is 27.

Let R represent Ruben's favourite number. Ruben input the numbers $(6,\ 4,\ R,\ 7)$ into his program and the output was 4 times his favourite number, or $4R$. What is Ruben's favourite number?



Themes: Algebra, Number Sense



Problem of the Week
Problem C and Solution
Sum Program

Problem

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Solution

We could attempt this problem using trial and error, and perhaps we would even stumble on the correct answer. However, this may take a long time and is not an efficient method for solving the problem. Instead we will provide a more algebraic solution.

The list of numbers inputted into the program was $(6,\ 4,\ R,\ 7)$. After the adjacent numbers are added together, we obtain $(10,\ 4+R,\ R+7)$. After the adjacent numbers in this list are added together, we obtain $(10+4+R,\ 4+R+R+7)$, which is equal to $(14+R,\ 11+2R)$. After the adjacent numbers in this list are added together, we obtain $(14+R+11+2R)$ which is equal to $(25+3R)$. Since there is now one number in the list, this number should be equal to the output of the program, which is $4R$. Thus,

$$\begin{aligned} 25+3R &= 4R \\ 25+3R-3R &= 4R-3R \\ 25 &= R \end{aligned}$$

Therefore, Ruben's favourite number is 25.

Computational Thinking (C)

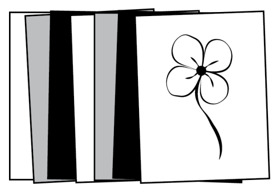
Take me to the cover

Problem of the Week

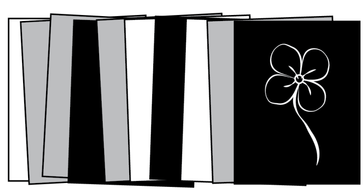
Problem C

Selling Sketches

Elouise sketches pictures and sells them in the park during a summer festival. She uses three colours of paper: white, grey, and black. First she sketches on white paper and puts it on the table for display. Then she sketches on grey paper, and then she sketches on black paper. After she has finished a sketch, she places it on top of the display pile. She continues in this pattern of white, grey, black, so that after completing seven sketches, her display pile would be as shown.



Elouise starts the festival with nothing in her display pile and continues sketching and adding pictures to the display pile throughout the festival. If someone wants to buy one of her sketches, they can only buy the sketch that is currently on the top of the pile. At the end of the festival, Elouise notes that she has sketched more pictures than she has sold, and the remaining sketches in her display pile are as shown.



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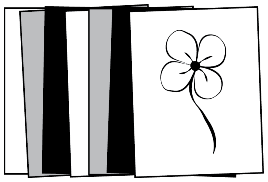
Themes: Algebra, Computational Thinking



Problem of the Week
Problem C and Solution
Selling Sketches

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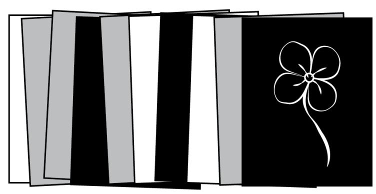

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Let W represent a sketch on white paper, G represent a sketch on grey paper, and B represent a sketch on black paper. Then the sequence in the display pile at the end of the festival is $WGGBGWBWGB$, where the leftmost sketches represent those on the bottom of the pile.

If no sketches were sold at all, the sequence would be $WGBWGBWGB\ldots$.

We will place a _ in our display pile sequence at the places where we know some sketches must have been sold, because it does not follow the sequence in which they were made. This gives $WG\_GB\_G\_W\_BWGB$. Moving from left to right, the fewest number of sketches that could fill the first blank is 2, namely BW. The fewest number of sketches that could fill the second blank is 1, namely W. The fewest number of sketches that could fill the third blank is 1, namely B. The fewest number of sketches that could fill the last blank is 1, namely G. Filling these in gives $WG\underline{BW}GB\underline{W}G\underline{B}W\underline{G}BWGB$, as desired.

Therefore, the minimum number of sketches that Elouise could have sold is $2 + 1 + 1 + 1 = 5$.

Problem of the Week

Problem C

Five on the ION

On Saturday afternoon, Ajay, Badr, Ciro, Dena, and Ema got on the same ION light rail train at University of Waterloo station. Each person was listening to something different on their headphones and each person got off at a different station. Using the clues below, determine which station each person got off at, and what they were listening to.

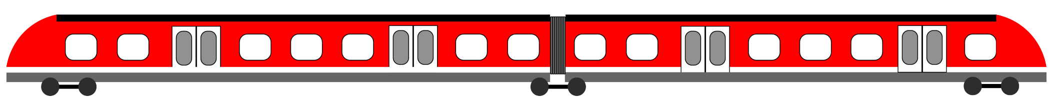
(1) Ciro got off the train at Fairway station but was *not* the person who was listening to the news.

(2) The person who got off at Central station was listening to an audiobook, and was *not* Dena.

(3) The person who was listening to a hockey game was *not* the person who got off at Borden station.

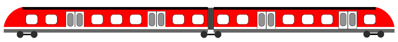
(4) Ema was listening to music and got off at Queen station.

(5) Ajay was listening to a podcast but was *not* the person who got off at Willis Way station.



Theme: Computational Thinking

Problem of the Week



Problem C and Solution

Five on the ION

Problem

On Saturday afternoon, Ajay, Badr, Ciro, Dena, and Ema got on the same ION light rail train at University of Waterloo station. Each person was listening to something different on their headphones and each person got off at a different station. Using the clues below, determine which station each person got off at, and what they were listening to.

(1) Ciro got off the train at Fairway station but was *not* the person who was listening to the news.

(2) The person who got off at Central station was listening to an audiobook, and was *not* Dena.

(3) The person who was listening to a hockey game was *not* the person who got off at Borden station.

(4) Ema was listening to music and got off at Queen station.

(5) Ajay was listening to a podcast but was *not* the person who got off at Willis Way station.

Solution

We can use a table to summarize the information in the clues. We will start by filling in the information from clues (1), (4), and (5). If we know that something does not go in a particular cell, we will also include that in the table.

Person	Station	Listening to
Ajay	*not* Willis Way	podcast
Badr		
Ciro	Fairway	*not* news
Dena		
Ema	Queen	music

From clue (2), we know the person who got off at Central station was listening to an audiobook. The only people who have not been matched with a station or what they were listening to are Badr and Dena. Since clue (2) also tells us this person was not Dena, it follows that Badr got off at Central station and was listening to an audiobook.

Then, since we know someone got off at Willis Way station and it was not Ajay, it follows that it must have been Dena since she's the only remaining person who hasn't been matched with a station.

Similarly, we know someone was listening to the news and it was not Ciro. Thus, it must have been Dena since she's the only remaining person who hasn't been matched with what they were listening to. These updates are shown in the table below.

Person	**Station**	**Listening to**
Ajay	*not* Willis Way	podcast
Badr	Central	audiobook
Ciro	Fairway	*not* news
Dena	Willis Way	news
Ema	Queen	music

Finally, clue (3) tells us someone was listening to a hockey game and someone got off at Borden station. Since Ciro is the only person who hasn't been matched with what they're listening to, Ciro must have been listening to a hockey game. Similarly since Ajay is the only person who hasn't been matched with a station, Ajay must have gotten off at Borden station. Thus, our final solution is shown.

Person	**Station**	**Listening to**
Ajay	Borden	podcast
Badr	Central	audiobook
Ciro	Fairway	hockey game
Dena	Willis Way	news
Ema	Queen	music

Problem of the Week
Problem C
Group Work

Piero has 7 tiles, each with a different integer from 1 to 7 written on it. There are many ways in which he can separate his tiles into groups, where each group contains at least one tile. For example, he can separate his tiles into 3 groups as shown.

1 2 4 6   3 7   5

The sum of the numbers in each of these groups is 13, 10, and 5, respectively.

Piero then separates his tiles into different groups and notices that the sum of the numbers in each group is the same. In how many different ways can Piero separate his tiles into at least two groups so that the sum of the numbers in each group is the same?

Themes: Computational Thinking, Number Sense

Problem of the Week

Problem C and Solution

Group Work

Problem

Piero has 7 tiles, each with a different integer from 1 to 7 written on it. There are many ways in which he can separate his tiles into groups, where each group contains at least one tile. For example, he can separate his tiles into 3 groups as shown.

The sum of the numbers in each of these groups is 13, 10, and 5, respectively.

Piero then separates his tiles into different groups and notices that the sum of the numbers in each group is the same. In how many different ways can Piero separate his tiles into at least two groups so that the sum of the numbers in each group is the same?

Solution

The sum of the numbers on Piero's tiles is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. If the sum of the numbers in each group is the same, then this sum must be a factor of 28. The positive factors of 28 are 1, 2, 4, 7, 14, and 28. Since the number 7 must be in one of the groups, it follows that the sum of each group must be at least 7. Also since there must be at least 2 groups, then the sum of each group cannot be 28, because in that case there would be only 1 group. It follows that the groups could each have a sum of 7 or a sum of 14. We will look at these two cases.

Case 1: Each group has a sum of 7.
In this case, there would be $28 \div 7 = 4$ groups. The only way to separate the tiles into 4 groups, each with a sum of 7, is as $\{7\},\ \{1, 6\},\ \{2, 5\},\ \{3, 4\}$.

Case 2: Each group has a sum of 14.
In this case, there would be $28 \div 14 = 2$ groups. To help count the possible groups, we note that one of the two groups must contain the tile with the 7. The possibilities for the other tiles are as follows:

- If the group with the 7 also contains the 6, then the only possibility is $\{1, 6, 7\}$. The other group must then be $\{2, 3, 4, 5\}$.
- If the group with the 7 also contains the 5, then the only possibility is $\{2, 5, 7\}$. The other group must then be $\{1, 3, 4, 6\}$.

- If the group with the 7 also contains the 4, then the group could be either $\{3, 4, 7\}$ or $\{1, 2, 4, 7\}$. The other group would then be $\{1, 2, 5, 6\}$ or $\{3, 5, 6\}$, respectively.
- If the group with the 7 also contains the 3, 2, or 1, then it must also contain the 4, 5, or 6, since $7 + 3 + 2 + 1 = 13$, which is less than 14. So there are no additional cases.

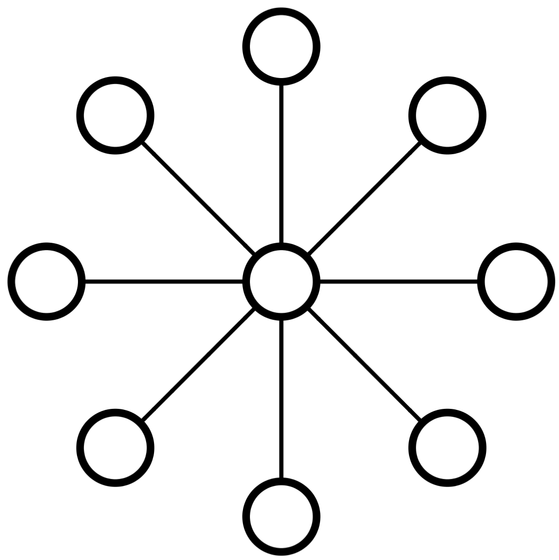
Thus, there are 4 ways to separate the tiles into 2 groups, each with a sum of 14.

Therefore, there are $1 + 4 = 5$ different ways in which Piero can separate his tiles into at least two groups so that the sum of the numbers in each group is the same. These are listed below.

- $\{7\},\ \{1, 6\},\ \{2, 5\},\ \{3, 4\}$
- $\{1, 6, 7\},\ \{2, 3, 4, 5\}$
- $\{2, 5, 7\},\ \{1, 3, 4, 6\}$
- $\{3, 4, 7\},\ \{1, 2, 5, 6\}$
- $\{1, 2, 4, 7\},\ \{3, 5, 6\}$

Problem of the Week
Problem C
The Snowflake Game

The Snowflake Game is played on a board that consists 9 circles. There is 1 circle in the centre of the board, with 4 line segments through it, and a circle at the end of each line segment.



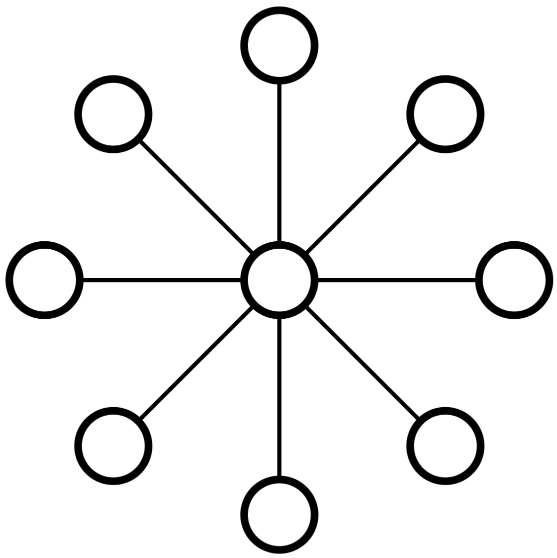
Two players alternate turns placing discs numbered 1 to 9 in the circles on the board. Each number can only be used once in any game. The object of the game is to be the first player to place a disc so that the sum of the three numbers along a line segment through the centre circle is exactly 15.

Alex and Blake play the game. Alex goes first. Show that Alex has a *winning strategy* if she places a 6 in the centre circle on her first turn. That is, show that if Alex places a 6 in the centre circle on her first turn, then no matter which numbers Blake plays, Alex can always win the game.

Problem of the Week
Problem C and Solution
The Snowflake Game

Problem

The Snowflake Game is played on a board that consists 9 circles. There is 1 circle in the centre of the board, with 4 line segments through it, and a circle at the end of each line segment.



Two players alternate turns placing discs numbered 1 to 9 in the circles on the board. Each number can only be used once in any game. The object of the game is to be the first player to place a disc so that the sum of the three numbers along a line segment through the centre circle is exactly 15.

Alex and Blake play the game. Alex goes first. Show that Alex has a *winning strategy* if she places a 6 in the centre circle on her first turn. That is, show that if Alex places a 6 in the centre circle on her first turn, then no matter which numbers Blake plays, Alex can always win the game.

Solution

If Alex places a 6 in the centre circle on her first turn, then the other two discs in the line segment would need to add to 9 in order for the sum to be 15.

Since each number can be used only once in a game, after Alex places the 6, the numbers that remain are 1, 2, 3, 4, 5, 7, 8, and 9.

- **Case 1:** Blake places a 1, 2, 4, 5, 7, or 8 on his first turn.

  Then there is always an unused number remaining that Alex can place on the same line segment to win the game on her second turn. That is, if Blake places a 1, then Alex should place an 8. If Blake places a 2, then Alex should place a 7. If Blake places a 4, then Alex should place a 5. If Blake places a 5, then Alex should place a 4. If Blake places a 7, then Alex should place a 2. If Blake places an 8, then Alex should place a 1.

- **Case 2:** Blake places a 3 on his first turn.

  Then the sum of the two discs on the line segment containing the 3 will be 9. Alex cannot win on her second turn since the only way to make the sum in that line segment 15 would be for her to place another 6. No number may be used more than once so this is not possible. However, if Alex completes the line segment by placing a 9 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7, and 8. Then, as in Case 1, no matter what Blake places on his second turn, there will always be a number that Alex can place on that same line segment so that the three numbers on the line segment sum to 15.

- **Case 3:** Blake places a 9 on his first turn.

  Then the sum of the two discs on the line segment containing the 9 will be 15. Alex cannot win on her second turn since playing any other disc on that line segment will make the sum greater than 15. However, if Alex completes the line segment by placing a 3 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7, and 8. Then, as in Case 1, no matter what Blake places on his second turn, there will always be a number that Alex can place on that same line segment so that the three numbers on the line segment sum to 15.

Therefore, we have shown that if Alex places a 6 in the centre circle on her first turn, then no matter which numbers Blake plays, Alex can always win the game on either her second or third turn.

For Further Thought:

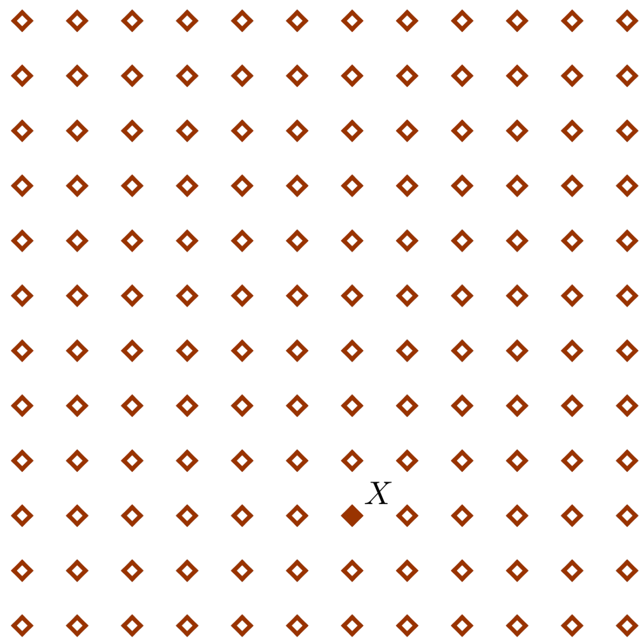
What other numbers can Alex place in the middle circle on her first turn to have a winning strategy? That is, what other numbers can Alex place in the middle circle on her first turn so that she can always win the game, no matter which numbers Blake plays?

Data Management (D)

Take me to the cover

Problem of the Week
Problem C
Diamond in the Rough

Using 144 diamonds, the 12 by 12 grid of diamonds below is created. One of the diamonds is coloured and labelled X.

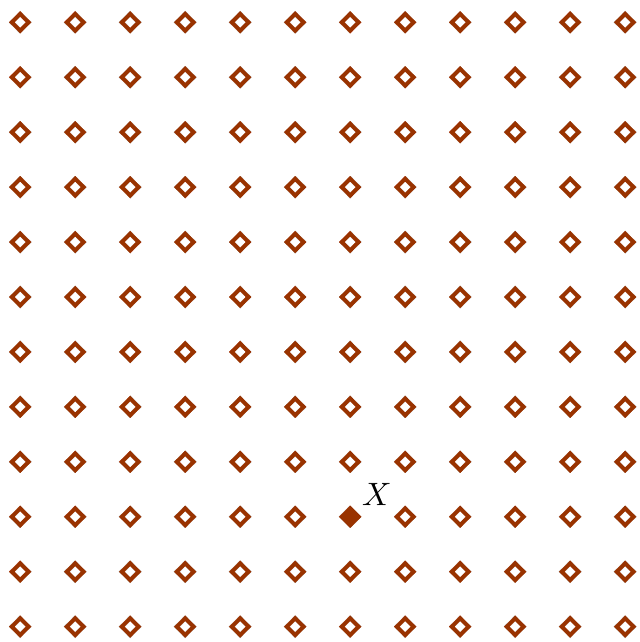


One of the other 143 diamonds in the grid is randomly chosen and is coloured in and labelled Y. What is the probability the line segment connecting X and Y is vertical or horizontal?

Problem of the Week
Problem C and Solution
Diamond in the Rough

Problem

Using 144 diamonds, the 12 by 12 grid of diamonds below is created. One of the diamonds is coloured and labelled X.



One of the other 143 diamonds in the grid is randomly chosen and is coloured in and labelled Y. What is the probability the line segment connecting X and Y is vertical or horizontal?

Solution

Line segment XY is vertical if Y is chosen from the diamonds in the column in which X lies. In this column there are 11 diamonds other than X which could be chosen to be Y so that XY is vertical.

Line segment XY is horizontal if Y is chosen from the diamonds in the row in which X lies. In this row there are 11 diamonds other than X which could be chosen to be Y so that XY is horizontal. Each of these 11 diamonds is different from the 11 diamonds in the column containing X. Thus, there are $11 + 11 = 22$ diamonds which may be chosen for Y so that XY is vertical or horizontal.

Since there are a total of 143 diamonds to choose Y from, the probability that Y is chosen so that XY is vertical or horizontal is $\frac{22}{143}$ or $\frac{2}{13}$.

Problem of the Week
Problem C
Moving Time

Stefania's family is looking for a new apartment and Stefania is hoping to get a dog after they move. Of the 65 apartments she has found online, 40 are close to a park, 28 allow dogs, and 8 are not close to a park and do not allow dogs. How many of the apartments are close to a park and allow dogs?

Themes: Algebra, Data Management

Problem of the Week
Problem C and Solution
Moving Time

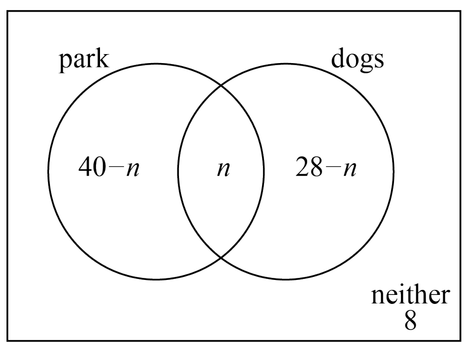
Problem

Stefania's family is looking for a new apartment and Stefania is hoping to get a dog after they move. Of the 65 apartments she has found online, 40 are close to a park, 28 allow dogs, and 8 are not close to a park and do not allow dogs. How many of the apartments are close to a park and allow dogs?

Solution

Solution 1

Let n be the number of apartments that are close to a park and allow dogs. Since 40 apartments are close to a park and n apartments are close to a park and allow dogs, then $40 - n$ apartments are close to a park but do not allow dogs. Similarly, since 28 apartments allow dogs and n apartments are close to a park and allow dogs, then $28 - n$ apartments allow dogs but are not close to a park. We also know that 8 apartments are not close to a park and do not allow dogs. This information is summarized in the following Venn diagram.



Since there are 65 apartments in total, then

$$\begin{aligned} 65 &= (40 - n) + n + (28 - n) + 8 \\ 65 &= 76 - n \\ n &= 76 - 65 = 11 \end{aligned}$$

Therefore, 11 of the apartments are close to a park and allow dogs.

Solution 2

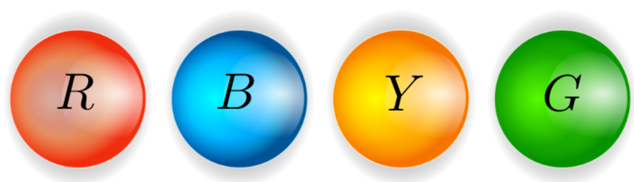
Since 8 of the 65 apartments are not close to a park and do not allow dogs, we know that $65 - 8 = 57$ apartments must be close to a park or allow dogs, or both. We know that the 40 apartments close to a park will include the apartments that are close to a park and also allow dogs. Similarly, the 28 apartments that allow dogs will include the apartments that are close to a park and also allow dogs. Thus, if we consider $40 + 28 = 68$, we will be double counting the apartments that are close to a park and also allow dogs. Since there are 57 apartments that are close to a park or allow dogs, or both, we have double-counted $68 - 57 = 11$ apartments. Thus, there must be 11 apartments that are both close to a park and allow dogs.

Problem of the Week
Problem C
Marbles, Marbles

A box contains 6 red marbles, 5 blue marbles, 2 yellow marbles, and 3 green marbles. Several orange marbles are added to the box. All the marbles in the box are identical except for colour.

A marble is then randomly selected from the box, and the probability that a blue or green marble is selected is $\frac{2}{7}$.

How many orange marbles were added to the box?



Themes: Algebra, Data Management

Problem of the Week

Problem C and Solution

Marbles, Marbles

Problem

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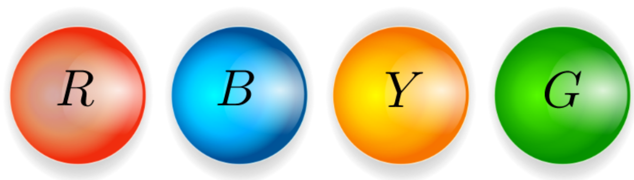


Solution

The number of blue and green marbles in the box is $5 + 3 = 8$.

Let n be the total number of marbles in the box after adding some orange marbles. Since the probability of picking a blue or green marble is $\frac{2}{7}$, we must have $\frac{8}{n} = \frac{2}{7}$.

If we multiply the numerator and denominator of the fraction $\frac{2}{7}$ by 4, we obtain $\frac{2}{7} = \frac{2\times 4}{7\times 4} = \frac{8}{28}$. Therefore, $\frac{8}{n} = \frac{8}{28}$. Since the fractions are equal and the numerators are equal, the denominators must also be equal. It follows that $n = 28$.

In the beginning, there were $5 + 6 + 3 + 2 = 16$ marbles in the box. Since there were 16 marbles in the box and there are now 28 marbles in the box, then $28 - 16 = 12$ orange marbles were added to the box.

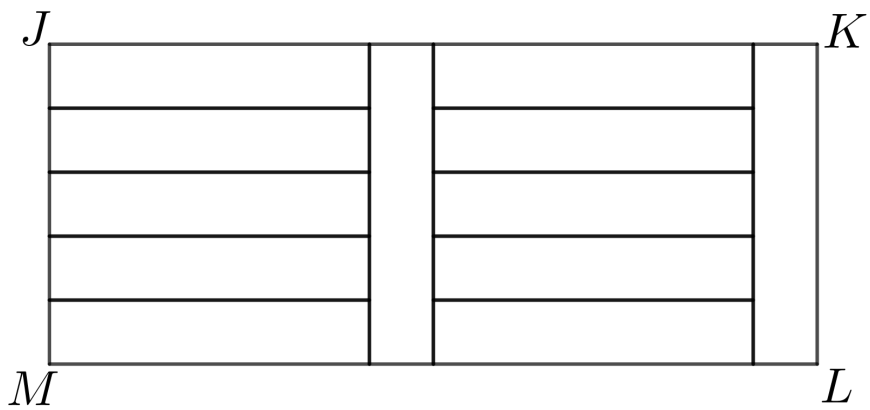
Therefore, 12 orange marbles were added to the box.

Geometry & Measurement (G)

Take me to the cover

Problem of the Week
Problem C
A Rectangle of Rectangles

Large rectangle $JKLM$ is formed by twelve identical smaller rectangles, as shown.



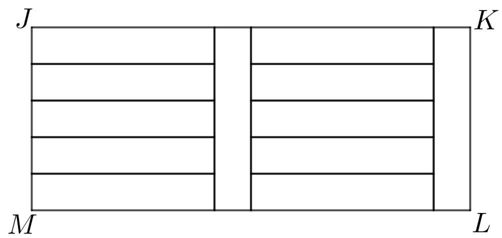
If the area of $JKLM$ is 540 cm^2, then determine the dimensions of the smaller rectangles.

Themes: Algebra, Geometry & Measurement

Problem of the Week
Problem C and Solution
A Rectangle of Rectangles

Problem

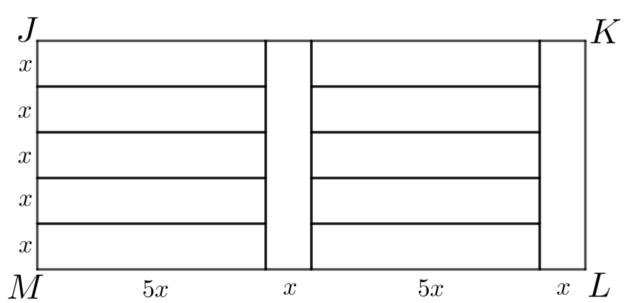
Large rectangle $JKLM$ is formed by twelve identical smaller rectangles, as shown.



If the area of $JKLM$ is 540 cm^2, then determine the dimensions of the smaller rectangles.

Solution

Let x be the width of one of the smaller identical rectangles, in cm. Five of the smaller rectangles are stacked on top of each other forming JM, so $JM = x + x + x + x + x = 5x$. Since $JKLM$ is a rectangle, $JM = KL = 5x$. Thus, $5x$ is also the length of a smaller rectangle. Therefore, a smaller rectangle is $5x$ cm by x cm.



From here, we proceed with two different solutions.

Solution 1

Since $JKLM$ is formed by twelve identical smaller rectangles, the area of rectangle $JKLM$ is equal to 12 times the area of one of the smaller rectangles.

$$\begin{aligned} \text{Area } JKLM &= 12 \times \text{Area of one smaller rectangle} \\ 540 &= 12 \times 5x \times x \\ 540 &= 60 \times x^2 \end{aligned}$$

Dividing both sides by 60, we obtain $x^2 = 9$. Since x is the width of a smaller rectangle, $x > 0$, and so $x = 3$ follows.

Thus, the width of a smaller rectangle is $x = 3$ cm and the length of a smaller rectangle is $5x = 5(3) = 15$ cm.

Therefore, the smaller rectangles are each 15 cm by 3 cm.

Solution 2

Side length ML is made up of the lengths of two of the smaller rectangles plus the widths of two of the smaller rectangles. Therefore, $ML = 5x + 5x + x + x = 12x$ and rectangle $JKLM$ is $12x$ cm by $5x$ cm.

To find the area of $JKLM$ we multiply the length ML by the width JM.

$$\begin{aligned} \text{Area } JKLM &= ML \times JM \\ 540 &= (12x) \times (5x) \\ 540 &= 12 \times 5 \times x \times x \\ 540 &= 60 \times x^2 \end{aligned}$$

Dividing both sides by 60, we obtain $x^2 = 9$. Since x is the width of a smaller rectangle, $x > 0$, and so $x = 3$ follows.

Thus, the width of a smaller rectangle is $x = 3$ cm and the length of a smaller rectangle is $5x = 5(3) = 15$ cm.

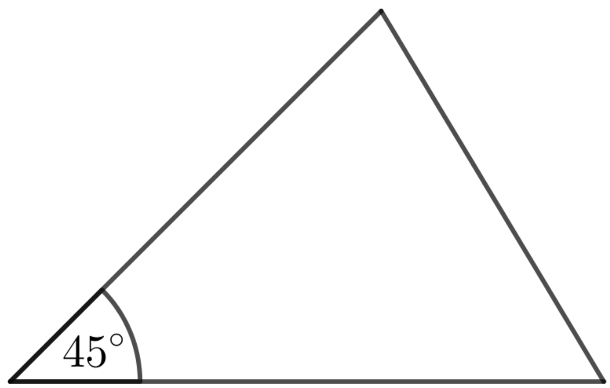
Therefore, the smaller rectangles are each 15 cm by 3 cm.

Problem of the Week

Problem C

Two Out of Three Angles

While measuring the angles in a triangle, Patricia found the measure of one of the angles is 45°. Once she had measured the other two angles, she noticed that the measures of these two angles are in the ratio $4 : 5$. What is the measure of each of the other two angles?



Themes: Algebra, Geometry & Measurement, Number Sense

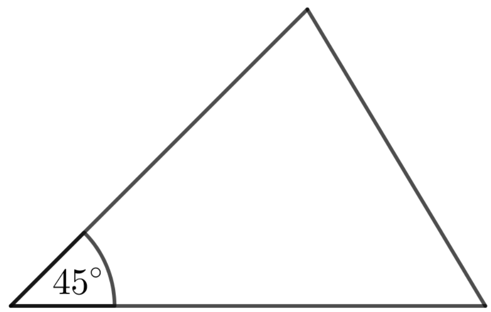
Problem of the Week

Problem C and Solution

Two Out of Three Angles

Problem

While measuring the angles in a triangle, Patricia found the measure of one of the angles is 45°. Once she had measured the other two angles, she noticed that the measures of these two angles are in the ratio $4:5$. What is the measure of each of the other two angles?



Solution

Solution 1

Since the measure of the unknown angles are in the ratio of $4:5$, then we can give the angles measures of $5n^\circ$ and $4n^\circ$.

Since the sum of the measures of the three angles in any triangle is 180°, then

$$\begin{aligned} 45 + 4n + 5n &= 180 \\ 4n + 5n &= 135 \\ 9n &= 135 \\ n &= 15 \end{aligned}$$

Therefore, the other two angles are $5n = 5(15) = 75^\circ$ and $4n = 4(15) = 60^\circ$.

Solution 2

Since the sum of the measures of the three angles in any triangle is 180°, then the sum of the measures of the two unknown angles in the triangle is $180^\circ - 45^\circ = 135^\circ$.

The measures of the two unknown angles are in the ratio $4:5$, and so one of the two angles measures $\frac{5}{4+5} = \frac{5}{9}$ of the sum of the two angles, while the other angle measures $\frac{4}{4+5} = \frac{4}{9}$ of the sum of the two angles.

That is, the larger of the two unknown angles measures $\frac{5}{9} \times 135^\circ = 75^\circ$, and the smaller of the unknown angles measures $\frac{4}{9} \times 135^\circ = 60^\circ$.

Therefore, the other two angles are 75° and 60°.

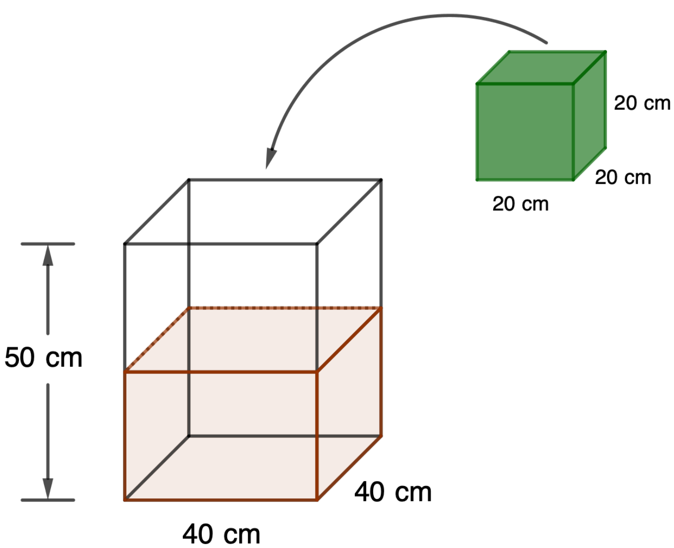
Problem of the Week

Problem C

Reach for the Top

Priya has a container in the shape of a rectangular prism with base 40 cm by 40 cm and height 50 cm. She fills the container with water so that the water reaches half of the height of the container. A solid cube with side length 20 cm is then placed in the container.

How far from the top of the container does the water now reach?



Themes: Geometry & Measurement, Number Sense

Problem of the Week

Problem C and Solution

Reach for the Top

Problem

Priya has a container in the shape of a rectangular prism with base 40 cm by 40 cm and height 50 cm. She fills the container with water so that the water reaches half of the height of the container. A solid cube with side length 20 cm is then placed in the container.

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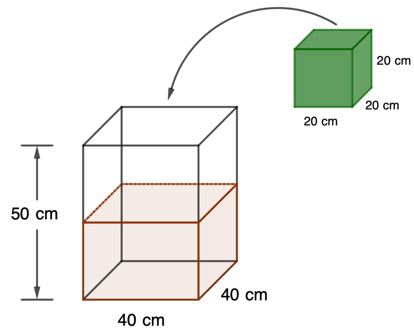


Solution

Solution 1

The height of the water is $\frac{50}{2} = 25$ cm. Since volume = length × width × height, the volume of water in the container is $40 \times 40 \times 25 = 40\,000$ cm^3.

The volume of the solid cube is $20 \times 20 \times 20 = 8000$ cm^3.

Thus, the total volume is $40\,000 + 8000 = 48\,000$ cm^3.

Let x represent the height of the water, in centimetres, after the cube is added. Then, using the formula for volume of a rectangular solid with 40 cm by 40 cm base, we have

$$\begin{aligned} 48\,000 &= 40 \times 40 \times x \\ 48\,000 &= 1600 \times x \\ x &= 30 \end{aligned}$$

Therefore, the new water height is 30 cm and the water is $50 - 30 = 20$ cm from the top of the container.

Solution 2

Let h be the height of a rectangular prism with base 40 cm by 40 cm and with the same volume as the solid cube. Since volume = length × width × height, we have

$$\begin{aligned} 40 \times 40 \times h &= 20 \times 20 \times 20 \\ 1600 \times h &= 8000 \\ h &= 5 \end{aligned}$$

Therefore, increasing the height of water in the container by 5 cm will increase the volume by 8000 cm^3, which is equal to the volume of the solid cube.

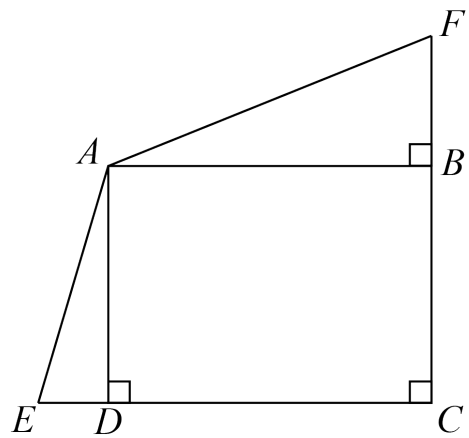
Therefore, the new water height is $25 + 5 = 30$ cm and the water is $50 - 30 = 20$ cm from the top of the container.

Problem of the Week

Problem C

Around the Outside

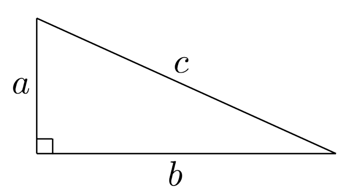
Two line segments, CE and CF, are perpendicular to each other, each with length 10. Rectangle $ABCD$ is drawn so that D is on CE, B is on CF with $BF = 4$, and the diagonal of $ABCD$ has length 10. Line segments EA and AF are then drawn. Determine the perimeter of quadrilateral $AFCE$, rounded to one decimal place.



Note: You may find the following useful:

The *Pythagorean Theorem* states, "In a right-angled triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides."

For example, if c is the length of the hypotenuse, and a and b are the lengths of the other two sides, then $c^2 = a^2 + b^2$.

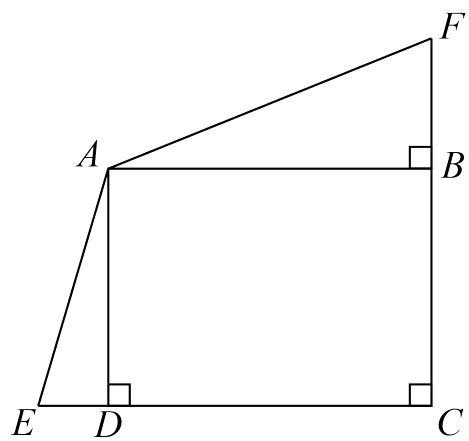


Theme: Geometry & Measurement

Problem of the Week
Problem C and Solution
Around the Outside

Problem

Two line segments, CE and CF, are perpendicular to each other, each with length 10. Rectangle $ABCD$ is drawn so that D is on CE, B is on CF with $BF = 4$, and the diagonal of $ABCD$ has length 10. Line segments EA and AF are then drawn. Determine the perimeter of quadrilateral $AFCE$, rounded to one decimal place.



NOTE: You may find the following useful:

The *Pythagorean Theorem* states, "In a right-angled triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides."

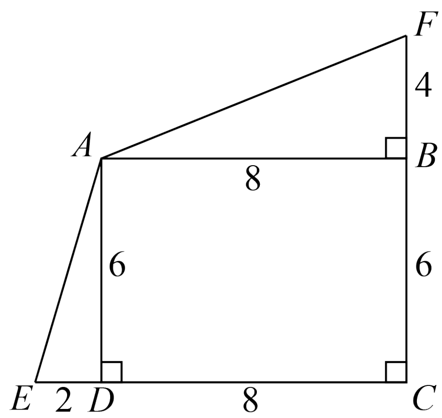
For example, if c is the length of the hypotenuse, and a and b are the lengths of the other two sides, then $c^2 = a^2 + b^2$.

Solution

First, $BC = CF - BF = 10 - 4 = 6$. Since $ABCD$ is a rectangle, $AD = BC = 6$. We then use the Pythagorean Theorem in $\triangle ABC$.

$$\begin{aligned} AB^2 &= AC^2 - BC^2 \\ &= 10^2 - 6^2 \\ &= 100 - 36 \\ &= 64 \end{aligned}$$

Therefore $AB = 8$, since $AB > 0$. Since $ABCD$ is a rectangle, $CD = AB = 8$. Then $DE = CE - CD = 10 - 8 = 2$. These lengths are shown on the diagram.



We then use the Pythagorean Theorem in $\triangle ADE$.

$$\begin{aligned} AE^2 &= AD^2 + DE^2 \\ &= 6^2 + 2^2 \\ &= 36 + 4 \\ &= 40 \end{aligned}$$

Therefore $AE = \sqrt{40}$, since $AE > 0$.

We then use the Pythagorean Theorem in $\triangle ABF$.

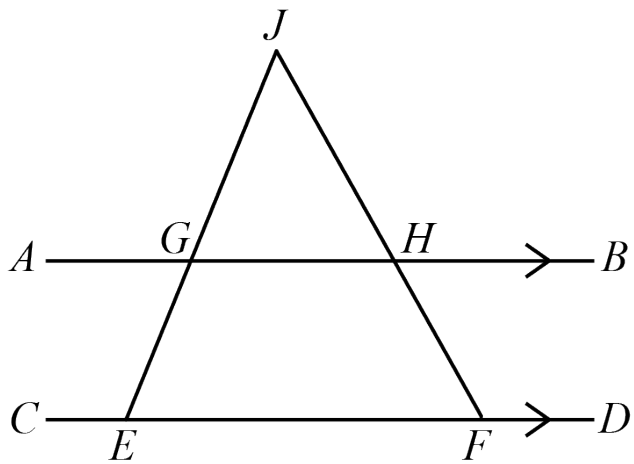
$$\begin{aligned} AF^2 &= AB^2 + BF^2 \\ &= 8^2 + 4^2 \\ &= 64 + 16 \\ &= 80 \end{aligned}$$

Therefore $AF = \sqrt{80}$, since $AF > 0$.

Thus, the perimeter of $AFCE$ is equal to
$AF + CF + CE + AE = \sqrt{80} + 10 + 10 + \sqrt{40} \approx 35.3$.

Problem of the Week
Problem C
Angled

Line segments AB and CD are parallel, with AB above CD. Point J lies above AB, and points E and F lie on CD, with E to the left of F, so that JE intersects AB at G and JF intersects AB at H.

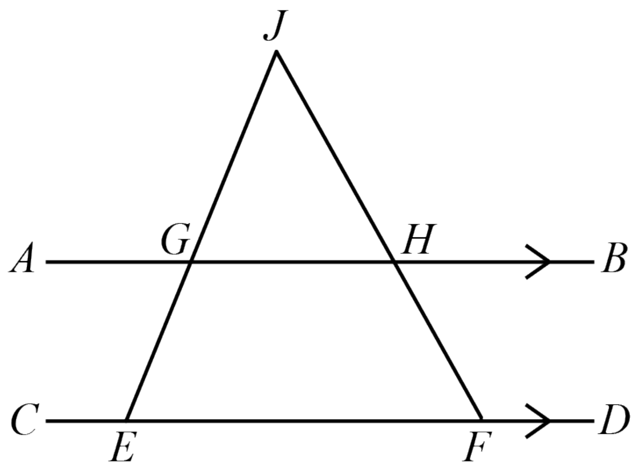


If $\angle CEG = 110^\circ$ and $\angle GHF = 122^\circ$, determine the measure of $\angle GJH$.

Problem of the Week
Problem C and Solution
Angled

Problem

Line segments AB and CD are parallel, with AB above CD. Point J lies above AB, and points E and F lie on CD, with E to the left of F, so that JE intersects AB at G and JF intersects AB at H.



If $\angle CEG = 110°$ and $\angle GHF = 122°$, determine the measure of $\angle GJH$.

Solution

Solution 1

Since JHF is a straight line, then $\angle JHG = 180° - \angle GHF = 180° - 122° = 58°$. Since AB and CD are parallel, $\angle AGJ = \angle CEG = 110°$. Since AGH is a straight line, $\angle JGH = 180° - \angle AGJ = 180° - 110° = 70°$. Since the three angles in a triangle add to $180°$, then

$$\angle GJH = 180° - \angle JGH - \angle JHG = 180° - 70° - 58° = 52°.$$

Solution 2

Since JHF is a straight line, then $\angle JHG = 180° - \angle GHF = 180° - 122° = 58°$. Since AB and CD are parallel, $\angle JFE = \angle JHG = 58°$. Since CEF is a straight line, $\angle JEF = 180° - \angle CEG = 180° - 110° = 70°$. Since the three angles in a triangle add to $180°$, then

$$\angle GJH = \angle EJF = 180° - \angle JEF - \angle JFE = 180° - 70° - 58° = 52°.$$

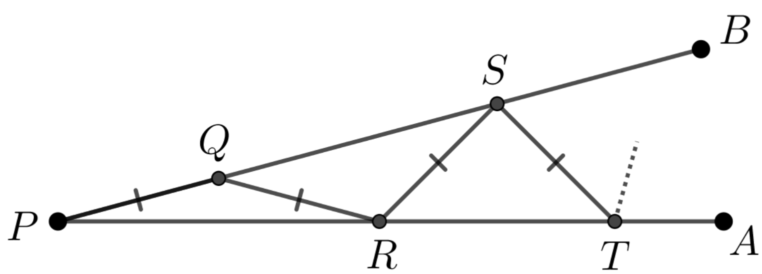
NOTE: Since $\triangle GJH$ and $\triangle EJF$ have all three angles in common, we can say that they are *similar triangles*. Similar triangles have properties that make them very useful in geometry problems.

Problem of the Week

Problem C

Back and Forth

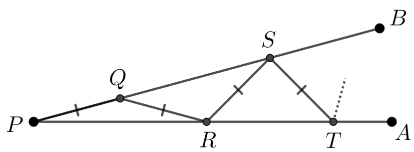
Line segments BP and AP are such that $\angle BPA = 12^\circ$. Points Q, R, S, T, ... alternate from one arm of the angle to the other, with Q on BP and R on AP, such that each point is located farther away from P than the point before, and $PQ = QR = RS = ST = \ldots$.



This creates isosceles triangles $\triangle PQR$, $\triangle QRS$, $\triangle RST$, and so on. Eventually, one of the isosceles triangles will also be an equilateral triangle.

How many isosceles triangles will be created before the equilateral triangle is created?

Theme: Geometry & Measurement



Problem of the Week

Problem C and Solution

Back and Forth

Problem

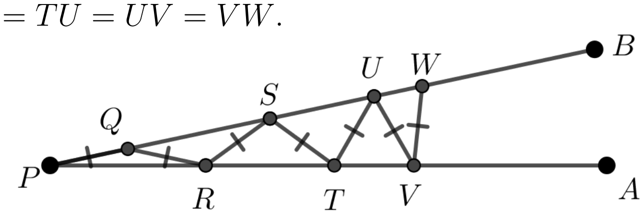
Line segments BP and AP are such that $\angle BPA = 12^\circ$. Points Q, R, S, T, $\ldots$ alternate from one arm of the angle to the other, with Q on BP and R on AP, such that each point is located farther away from P than the point before, and $PQ = QR = RS = ST = \ldots$. This creates isosceles triangles $\triangle PQR$, $\triangle QRS$, $\triangle RST$, and so on. Eventually, one of the isosceles triangles will also be an equilateral triangle.

How many isosceles triangles will be created before the equilateral triangle is created?

Solution

We will show that there will be 4 isosceles triangles created before the equilateral triangle is created. Throughout this solution there will be references to the Exterior Angle Theorem, which states: "The measure of an exterior angle of a triangle is equal to the sum of the measures of the opposite interior angles." For example, $\angle SQR$ is exterior to $\triangle QPR$ and $\angle SQR = \angle QPR + \angle QRP$.

We extend the diagram to include U on BP, V on AP, and W on BP such that $PQ = QR = RS = ST = TU = UV = VW$.



In $\triangle QPR$, $\angle QPR = 12^\circ$ and $PQ = QR$. Therefore, $\triangle QPR$ is isosceles and $\angle QRP = \angle QPR = 12^\circ$. Since $\angle SQR$ is exterior to $\triangle QPR$, by the Exterior Angle Theorem, $\angle SQR = \angle QPR + \angle QRP = 12^\circ + 12^\circ = 24^\circ$.

In $\triangle RQS$, $\angle SQR = 24^\circ$ and $QR = RS$. Therefore, $\triangle RQS$ is isosceles and $\angle RSQ = \angle SQR = 24^\circ$. Since $\angle SRT$ is exterior to $\triangle PRS$, by the Exterior Angle Theorem, $\angle SRT = \angle SPR + \angle PSR = 12^\circ + 24^\circ = 36^\circ$.

In $\triangle SRT$, $\angle SRT = 36^\circ$ and $SR = ST$. Therefore, $\triangle SRT$ is isosceles and $\angle STR = \angle SRT = 36^\circ$. Since $\angle UST$ is exterior to $\triangle PST$, by the Exterior Angle Theorem, $\angle UST = \angle SPT + \angle STP = 12^\circ + 36^\circ = 48^\circ$.

In $\triangle TSU$, $\angle UST = 48^\circ$ and $ST = TU$. Therefore, $\triangle TSU$ is isosceles and $\angle TUS = \angle UST = 48^\circ$. Since $\angle UTV$ is exterior to $\triangle PUT$, by the Exterior Angle Theorem, $\angle UTV = \angle UPT + \angle PUT = 12^\circ + 48^\circ = 60^\circ$.

In $\triangle UTV$, $\angle UTV = 60^\circ$ and $TU = UV$. Therefore, $\triangle UTV$ is isosceles and $\angle UVT = \angle UTV = 60^\circ$. Since $\angle UVT = \angle UTV = 60^\circ$, the remaining angle in $\triangle UTV$ is $\angle TUV = 180^\circ - 60^\circ - 60^\circ = 60^\circ$. Since all three angles equal 60°, $\triangle UTV$ is equilateral.

Since $\triangle UTV$ is the fifth isosceles triangle created, there are 4 isosceles triangles created before the equilateral triangle is created.

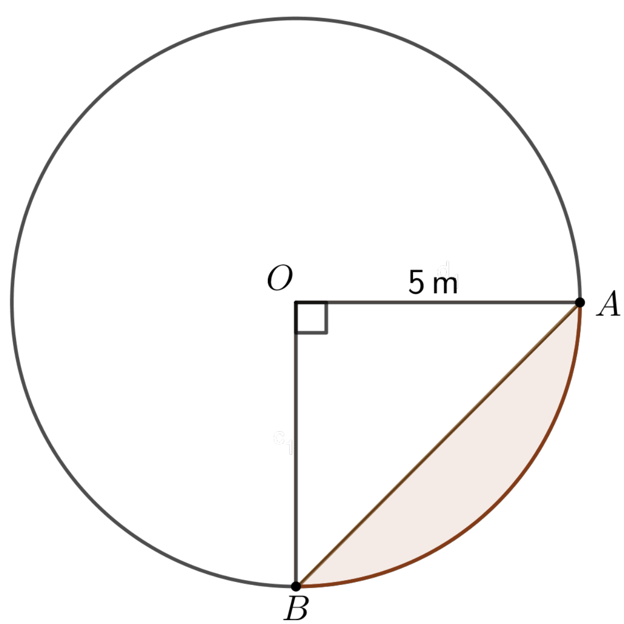
Extension: In our problem, we started with $\angle APB = 12^\circ$ and we eventually created an equilateral triangle. What other values for $\angle APB$ will eventually create an equilateral triangle?

Problem of the Week
Problem C
A Wet Sidewalk

A sprinkler sprays water in a circular region with a radius of 5 m. It is positioned in such a way that some of the water is landing on a nearby sidewalk.

In the diagram, the circle with centre O represents the circular region covered by the water sprinkler. Points A and B lie on the circle with $\angle AOB = 90^\circ$ and $OA = 5$ m. The shaded region, which is the region inside sector AOB but outside of $\triangle AOB$, represents the region of the sidewalk where water is landing.

Determine the area of the region of the sidewalk where the water is landing, correct to one decimal place.



Theme: Geometry & Measurement

Problem of the Week
Problem C and Solution
A Wet Sidewalk

Problem

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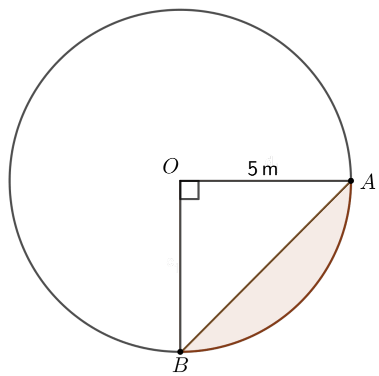
Determine the area of the region of the sidewalk where the water is landing, correct to one decimal place.



Solution

$\triangle AOB$ along with the shaded region cover $\frac{1}{4}$ of the area of the circle covered by the sprinkler. To determine the area of the shaded region, we need to find the area of the triangle and subtract it from one-quarter of the area of the circle.

Since $\angle AOB = 90^\circ$, we can use OA as the base and OB as the height in the formula for the area of a triangle to find the area of $\triangle AOB$. Further, since OA and OB are radii of the circle, we know that $OA = OB = 5$ m. Therefore,

$$\text{Area } \triangle AOB = \frac{\text{base} \times \text{height}}{2} = \frac{OA \times OB}{2} = \frac{5 \times 5}{2} = \frac{25}{2} = 12.5 \text{ m}^2$$

To determine the area of the quarter circle, we will use the formula for the area of a circle, $A = \pi r^2$, and then divide the result by 4. That is,

$$\text{Area of the Quarter Circle} = \frac{\pi \times r^2}{4} = \frac{\pi \times 5 \times 5}{4} = \frac{\pi \times 25}{4} = (6.25 \times \pi) \text{ m}^2$$

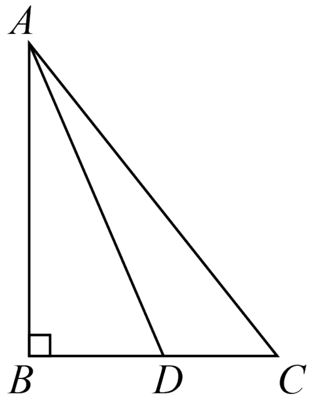
We can now determine the area of the shaded region.

$$\begin{aligned}\text{Area of Shaded Region} &= \text{Area of the Quarter Circle} - \text{Area } \triangle AOB \\ &= 6.25 \times \pi - 12.5 \\ &\approx 7.1 \text{ m}^2\end{aligned}$$

Therefore, the area of the shaded region, correct to one decimal place, is 7.1 m^2.

Problem of the Week
Problem C
Three Triangles

The right-angled triangle $\triangle ABC$ has $\angle ABC = 90^\circ$ and $AB = 12$. Point D is on side BC such that $BD = 5$ and the area of $\triangle ADC$ is 80% of the area of $\triangle ABD$.

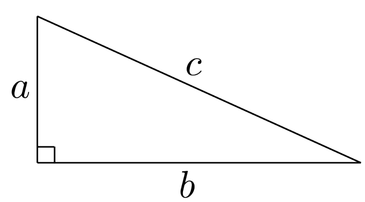


Determine the perimeter of $\triangle ADC$.

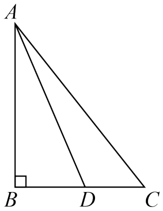
NOTE: You may find the following useful:

The *Pythagorean Theorem* states, "In a right-angled triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides."

In the right-angled triangle shown, c is the hypotenuse, a and b are the lengths of the other two sides, and $c^2 = a^2 + b^2$.



THEME: GEOMETRY & MEASUREMENT



Problem of the Week

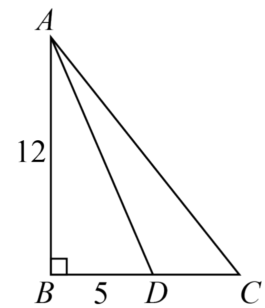
Problem C and Solution

Three Triangles

Problem

The right-angled triangle $\triangle ABC$ has $\angle ABC = 90^\circ$ and $AB = 12$. Point D is on side BC such that $BD = 5$ and the area of $\triangle ADC$ is 80% of the area of $\triangle ABD$. Determine the perimeter of $\triangle ADC$.

Solution



We start by labeling our diagram with the given information. We then notice that since AB is perpendicular to BC, it follows that AB is perpendicular to DC. Thus, if the base of $\triangle ADC$ is DC, then its height is AB.

To find the area of a triangle, multiply the length of the base by the height and divide by 2. Therefore,

$$\text{area of } \triangle ABD = BD \times AB \div 2 = 5 \times 12 \div 2 = 30.$$

Since the area of $\triangle ADC$ is 80% of the area of $\triangle ABD$, the area of $\triangle ADC$ is equal to $0.8 \times 30 = 24$.

We also know the area of $\triangle ADC$ is equal to $DC \times AB \div 2$. Thus,

$$\begin{aligned} DC \times AB \div 2 &= 24 \\ DC \times 12 \div 2 &= 24 \\ DC \times 6 &= 24 \\ DC &= 24 \div 6 = 4 \end{aligned}$$

Thus, $BC = BD + DC = 5 + 4 = 9$. Using the Pythagorean Theorem in $\triangle ABC$,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 12^2 + 9^2 \\ &= 225 \end{aligned}$$

Thus, $AC = \sqrt{225} = 15$, since $AC > 0$.

Using the Pythagorean Theorem in $\triangle ABD$,

$$\begin{aligned} AD^2 &= AB^2 + BD^2 \\ &= 12^2 + 5^2 \\ &= 169 \end{aligned}$$

Thus, $AD = \sqrt{169} = 13$, since $AD > 0$.

Therefore, the perimeter of $\triangle ADC$ is $AD + DC + AC = 13 + 4 + 15 = 32$.

Problem of the Week
Problem C
Efficient Tiling

Sophia has an unlimited supply of square tiles. Sophia has 1 cm by 1 cm tiles, 2 cm by 2 cm tiles, 3 cm by 3 cm tiles, and so on. Every tile has integer side lengths, in centimetres.

A rectangular bathroom floor with an 84 cm by 140 cm surface is to be completely covered by identical square tiles, none of which can be cut. Sophia knows that the floor can be completely covered with 11 760 tiles of size 1 cm by 1 cm, since $84 \times 140 = 11\,760$ cm^2. However, Sophia wants to use the minimum number of identical sized tiles to complete the job.

Determine the minimum number of identical sized tiles required to completely cover the bathroom floor.

Themes: Geometry & Measurement, Number Sense

Problem of the Week
Problem C and Solution
Efficient Tiling

Problem

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A rectangular bathroom floor with an 84 cm by 140 cm surface is to be completely covered by identical square tiles, none of which can be cut. Sophia knows that the floor can be completely covered with 11 760 tiles of size 1 cm by 1 cm, since $84 \times 140 = 11\,760$ cm^2. However, Sophia wants to use the minimum number of identical sized tiles to complete the job.

Determine the minimum number of identical sized tiles required to completely cover the bathroom floor.

Solution

To use the least number of tiles, Sophia must use the largest tile possible. The square tile must have sides less than or equal to 84 cm. If the tile has side length greater than 84 cm, it would have to be cut to fit the width of the floor.

Since the tiles are square and must completely cover the floor, the side length of the tile must be a number that is a factor of both 84 and 140. In fact, since Sophia wants the largest side length, she is looking for the greatest common factor of 84 and 140.

The positive factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84.

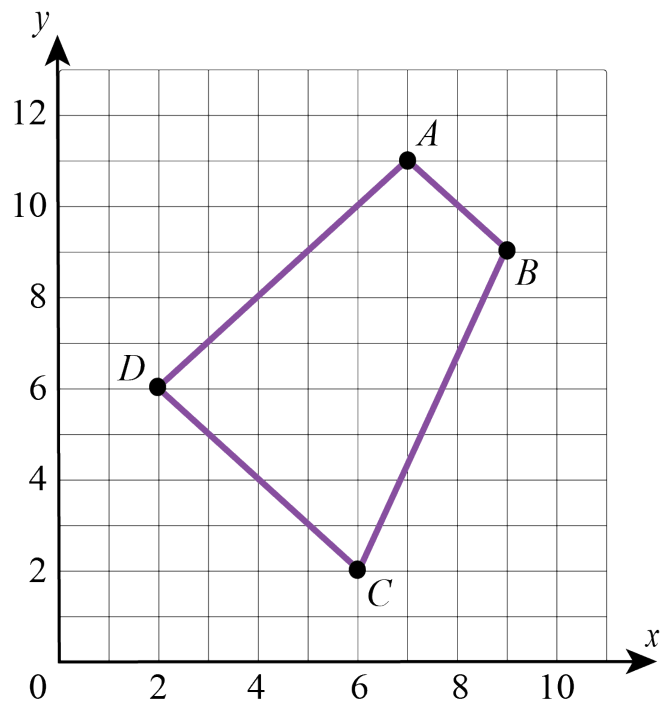
The positive factors of 140 are 1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70, and 140.

The largest number common to both lists is 28. Therefore, the greatest common factor of 84 and 140 is 28. The required tiles are 28 cm by 28 cm. Since $84 \div 28 = 3$, the surface is 3 tiles wide. Since $140 \div 28 = 5$, the surface is 5 tiles long. The minimum number of tiles required is $3 \times 5 = 15$ tiles.

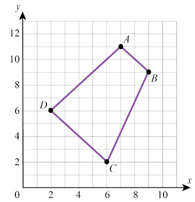
The number of 28 cm by 28 cm tiles required to cover the floor area is 15. This is the minimum number of tiles required.

Problem of the Week
Problem C
Irregular Area

Points $A(7, 11)$, $B(9, 9)$, $C(6, 2)$, and $D(2, 6)$ form quadrilateral $ABCD$. Determine the area of $ABCD$.



Theme: Geometry & Measurement



Problem of the Week
Problem C and Solution
Irregular Area

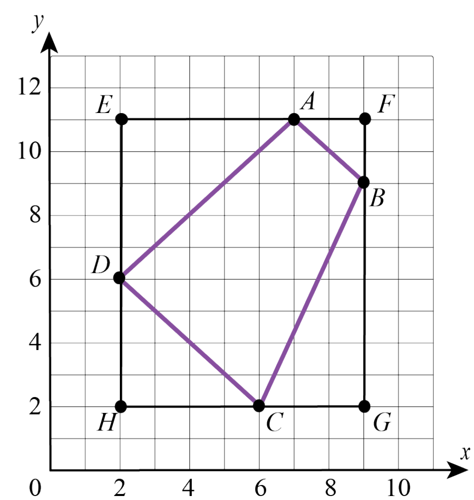
Problem

Points $A(7, 11)$, $B(9, 9)$, $C(6, 2)$, and $D(2, 6)$ form quadrilateral $ABCD$. Determine the area of $ABCD$.

Solution

Solution 1

Consider points $E(2, 11)$, $F(9, 11)$, $G(9, 2)$, and $H(2, 2)$. Draw in $EFGH$.



Since E and F both have y-coordinate 11, EF is a horizontal line which passes through A.

Since F and G both have x-coordinate 9, FG is a vertical line which passes through B.

Since G and H both have y-coordinate 2, GH is a horizontal line which passes through C.

Since E and H both have x-coordinate 2, EH is a vertical line which passes through D.

Thus, $EFGH$ is a rectangle that encloses $ABCD$. Also,

$$\text{area } ABCD = \text{area } EFGH - \text{area } \triangle AED - \text{area } \triangle AFB - \text{area } \triangle BGC - \text{area } \triangle CHD$$

In rectangle $EFGH$, $EF = 9 - 2 = 7$ and $EH = 11 - 2 = 9$. The area of rectangle $EFGH = EF \times EH = 7 \times 9 = 63$ units^2.

Since $EFGH$ is a rectangle, $\triangle AED$ is right-angled at E. Since $AE = 7 - 2 = 5$ and $ED = 11 - 6 = 5$, the area of $\triangle AED = \frac{AE \times ED}{2} = \frac{5 \times 5}{2} = \frac{25}{2} = 12.5$ units^2.

Since $EFGH$ is a rectangle, $\triangle AFB$ is right-angled at F. Since $FA = 9 - 7 = 2$ and $FB = 11 - 9 = 2$, the area of $\triangle AFB = \frac{FA \times FB}{2} = \frac{2 \times 2}{2} = \frac{4}{2} = 2$ units^2.

Since $EFGH$ is a rectangle, $\triangle BGC$ is right-angled at G. Since $GC = 9 - 6 = 3$ and $BG = 9 - 2 = 7$, the area of $\triangle BGC = \frac{GC \times BG}{2} = \frac{3 \times 7}{2} = \frac{21}{2} = 10.5$ units^2.

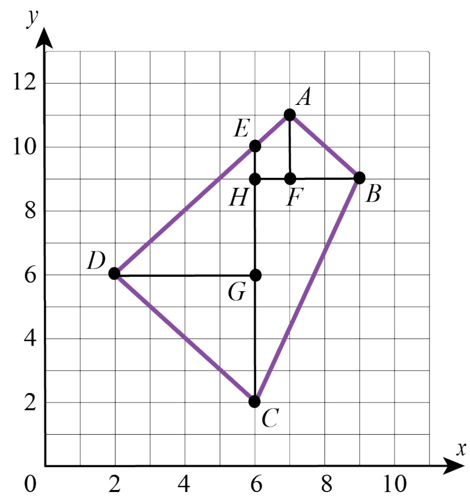
Since $EFGH$ is a rectangle, $\triangle CHD$ is right-angled at H. Since $CH = 6 - 2 = 4$ and $DH = 6 - 2 = 4$, the area of $\triangle CHD = \frac{CH \times DH}{2} = \frac{4 \times 4}{2} = \frac{16}{2} = 8$ units^2.

Therefore,

$$\begin{aligned}\text{area } ABCD &= \text{area } EFGH - \text{area } \triangle AED - \text{area } \triangle AFB - \text{area } \triangle BGC - \text{area } \triangle CHD \\ &= 63 - 12.5 - 2 - 10.5 - 8 \\ &= 30 \text{ units}^2\end{aligned}$$

Solution 2

Consider points $E(6,10)$, $F(7,9)$, $G(6,6)$, and $H(6,9)$. Draw in AF, EC, HB, and DG.



Since A and F both have x-coordinate 7, AF is a vertical line.

Since E, H, G, and C have x-coordinate 6, EC is a vertical line which passes through H and G.

Since H, F, and B have y-coordinate 9, HB is a horizontal line which passes through F.

Since D and G both have y-coordinate 6, DG is a horizontal line.

To determine the area of $ABCD$, we find the areas in the interior shapes, $\triangle AFB$, $\triangle BHC$, $\triangle DGC$, $\triangle DGE$, and trapezoid $EAFH$, and calculate their sum.

Since AF is vertical and HB is horizontal and passes through F, $\triangle AFB$ is right-angled at F. Since $AF = 11 - 9 = 2$ and $FB = 9 - 7 = 2$, area $\triangle AFB = \frac{AF \times FB}{2} = \frac{2\times 2}{2} = \frac{4}{2} = 2$ units^2.

Since HB is horizontal and EC is vertical and passes through H, $\triangle BHC$ is right-angled at H. Since $HB = 9 - 6 = 3$ and $HC = 9 - 2 = 7$, area $\triangle BHC = \frac{HB \times HC}{2} = \frac{3\times 7}{2} = \frac{21}{2} = 10.5$ units^2.

Since DG is horizontal and EC is vertical and passes through G, $\triangle DGC$ is right-angled at G. Since $DG = 6 - 2 = 4$ and $GC = 6 - 2 = 4$, area $\triangle DGC = \frac{DG \times GC}{2} = \frac{4\times 4}{2} = \frac{16}{2} = 8$ units^2.

Since DG is horizontal and EC is vertical and passes through G, $\triangle DGE$ is right-angled at G. Since $DG = 6 - 2 = 4$ and $EG = 10 - 6 = 4$, area $\triangle DGE = \frac{DG \times EG}{2} = \frac{4\times 4}{2} = \frac{16}{2} = 8$ units^2.

Since trapezoid $EAFH$ has side lengths $EH = 10 - 9 = 1$ and $AF = 11 - 9 = 2$, and height $HF = 7 - 6 = 1$, area trapezoid $EAFH = \frac{EH + AF}{2} \times HF = \frac{1+2}{2} \times 1 = \frac{3}{2} \times 1 = 1.5$ units^2.

Therefore,

$$\begin{aligned}\text{area } ABCD &= \text{area } \triangle AFB + \text{area } \triangle BHC + \text{area } \triangle DGC + \text{area } \triangle DGE + \text{area trapezoid } \triangle EAFH \\ &= 2 + 10.5 + 8 + 8 + 1.5 \\ &= 30 \text{ units}^2\end{aligned}$$

Number Sense (N)

Take me to the cover

Problem of the Week
Problem C
Tankmates

Niall has three large fish tanks and would like to put some tetras and guppies in each tank. He currently has 19 tetras and 18 guppies and doesn't want any of his tanks to contain more guppies than tetras. Also, he would like each tank to have at least 5 tetras and 3 guppies.

Determine the largest number of fish that can be in one of his fish tanks.

Theme: Number Sense

Problem of the Week
Problem C and Solution
Tankmates

Problem

Niall has three large fish tanks and would like to put some tetras and guppies in each tank. He currently has 19 tetras and 18 guppies and doesn't want any of his tanks to contain more guppies than tetras. Also, he would like each tank to have at least 5 tetras and 3 guppies.

Determine the largest number of fish that can be in one of his fish tanks.

Solution

Since each fish tank must have at least 5 tetras and 3 guppies, we will start by placing 5 tetras and 3 guppies into each fish tank. The number of fish now in each tank is summarized in the table below.

	Tank 1	Tank 2	Tank 3
Number of tetras	5	5	5
Number of guppies	3	3	3

We have placed $5 \times 3 = 15$ tetras and $3 \times 3 = 9$ guppies, so we have $19 - 15 = 4$ tetras and $18 - 9 = 9$ guppies left to place.

In order to place the largest number of fish in a tank, as many as possible of the remaining fish should be placed in one particular tank. We will start by adding all of the remaining tetras to Tank 1. The number of fish now in each tank is summarized in the table below.

	Tank 1	Tank 2	Tank 3
Number of tetras	9	5	5
Number of guppies	3	3	3

Since Niall doesn't want any of his tanks to contain more guppies than tetras, Tank 1 can contain at most 9 guppies. Since we want Tank 1 to contain the largest number of fish, we will add 6 guppies to Tank 1.

Then we have $9 - 6 = 3$ guppies left to place. We cannot add them to Tank 1. However, we can add 2 guppies to Tank 2, and 1 guppy to Tank 3. The number of fish now in each tank is summarized in the table below.

	Tank 1	Tank 2	Tank 3
Number of tetras	9	5	5
Number of guppies	9	5	4

Therefore, the largest number of fish that can be in one of Niall's fish tanks is 18.

Problem of the Week
Problem C
They Take the Cake

Jessica, Callista, Peter, and Monica went to the bakery to buy seven cakes. Each cake costs \$9.00. Jessica paid \$27.00, Callista paid \$9.00, and Peter paid \$22.50. Monica paid the remaining amount. They divide the cakes so that the fraction of the total that each person paid is equal to the fraction of the total amount of cake that each person receives.

What amount of cake should each person receive?

Theme: Number Sense

Problem of the Week
Problem C and Solution
They Take the Cake

Problem

Jessica, Callista, Peter, and Monica went to the bakery to buy seven cakes. Each cake costs \$9.00. Jessica paid \$27.00, Callista paid \$9.00, and Peter paid \$22.50. Monica paid the remaining amount. They divide the cakes so that the fraction of the total that each person paid is equal to the fraction of the total amount of cake that each person receives.

What amount of cake should each person receive?

Solution

Solution 1

If Jessica, Callista, Peter and Monica buy a total of seven cakes at \$9.00 per cake, then they paid a total of $7 \times \$9.00 = \$63.00$.

Jessica paid \$27.00, Callista paid \$9.00, Peter paid \$22.50 and Monica paid the remainder. Therefore, Monica paid $\$63.00 - \$27.00 - \$9.00 - \$22.50 = \$4.50$.

Since $3 \times \$9 = \$27.00$ then Jessica should receive 3 cakes.

Since $1 \times \$9 = \$9.00$ then Callista should receive 1 cake.

Since $0.5 \times \$9 = \$4.50$ then Monica should receive 0.5 cakes.

Thus, Peter should get $7 - 3 - 1 - 0.5 = 2.5$ cakes.

Solution 2

If Jessica, Callista, Peter and Monica buy a total of seven cakes at \$9.00 per cake, then they paid a total of $7 \times \$9.00 = \$63.00$.

Jessica paid \$27.00, Callista paid \$9.00, Peter paid \$22.50 and Monica paid the remainder. Therefore, Monica paid $\$63.00 - \$27.00 - \$9.00 - \$22.50 = \$4.50$.

Jessica should receive $\frac{27}{63} = \frac{3}{7}$ of the total amount of cake. So Jessica should receive $\frac{3}{7}$ of 7, or $\frac{3}{7} \times 7 = 3$ cakes.

Callista should receive $\frac{9}{63} = \frac{1}{7}$ of the total amount of cake. So Callista should receive $\frac{1}{7}$ of 7, or $\frac{1}{7} \times 7 = 1$ cake.

Peter should receive $\frac{22.5}{63} = \frac{225}{630} = \frac{25}{70} = \frac{5}{14}$ of the total amount of cake. So Peter should receive $\frac{5}{14}$ of 7, or $\frac{5}{14} \times 7 = \frac{5}{2} = 2\frac{1}{2}$ cakes.

Monica should receive $\frac{4.50}{63} = \frac{45}{630} = \frac{5}{70} = \frac{1}{14}$ of the total amount of cake. So Monica should receive $\frac{1}{14}$ of 7, or $\frac{1}{14} \times 7 = \frac{1}{2}$ of a cake.

We can check that all of the cake has been distributed. If there were any errors in our solution, this may help us catch them. Jessica receives 3 cakes, Callista receives 1 cake, Peter receives $2\frac{1}{2}$ cakes and Monica receives $\frac{1}{2}$ of a cake. The total number of cakes distributed is $3 + 1 + 2\frac{1}{2} + \frac{1}{2} = 7$, as required.

Problem of the Week

Problem C

The Sequence of the Year

In a sequence of numbers, each number in the sequence is called a *term*. In the sequence 2, 4, 6, 8, the first term is 2, the second term is 4, the third term is 6, and the fourth term is 8.

In another sequence, the first term is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

What is the 2024$^{\text{th}}$ term in the sequence?

Themes: Algebra, Number Sense

Problem of the Week
Problem C and Solution
The Sequence of the Year

Problem

In a sequence of numbers, each number in the sequence is called a *term*. In the sequence 2, 4, 6, 8, the first term is 2, the second term is 4, the third term is 6, and the fourth term is 8.

In another sequence, the first term is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

What is the 2024^{th} term in the sequence?

Solution

We will begin by finding more terms in the sequence. The first 14 terms of the sequence are 24, 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1.

If we continue the sequence, we will see that the terms 4, 2, and 1 will continue to repeat. Thus, the 9^{th} term, 12^{th} term, 15^{th} term, and so on, will each have a value of 4. Notice that these term numbers are all multiples of 3. It follows that every term number after 9 that is a multiple of 3 will have a value of 4.

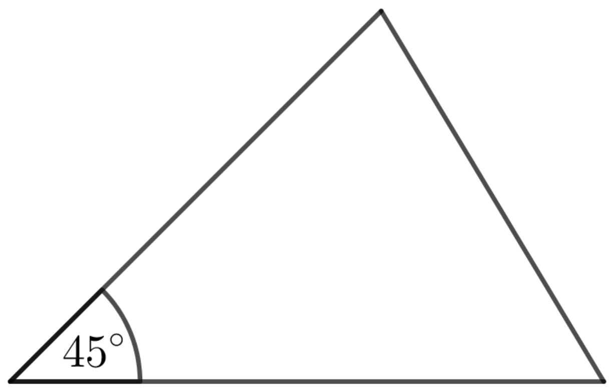
Thus, since 2022 is a multiple of 3, the 2022^{nd} term will have a value of 4. Then, the 2023^{rd} term will have a value of 2, and the 2024^{th} term will have a value of 1.

Extension:

In 1937, the mathematician Lothat Collatz wondered if any sequence whose terms after the first are determined in this way would always eventually reach the number 1, regardless of which number you started with. This problem is actually still unsolved today and is called the Collatz Conjecture.

Problem of the Week
Problem C
Two Out of Three Angles

While measuring the angles in a triangle, Patricia found the measure of one of the angles is 45°. Once she had measured the other two angles, she noticed that the measures of these two angles are in the ratio $4 : 5$. What is the measure of each of the other two angles?



Themes: Algebra, Geometry & Measurement, Number Sense

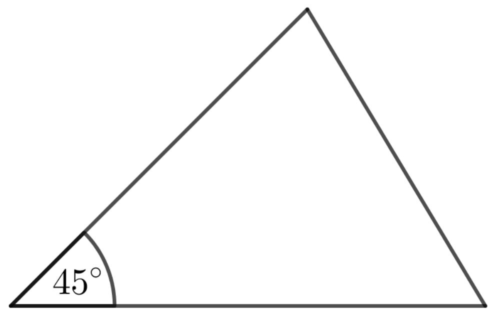
Problem of the Week

Problem C and Solution

Two Out of Three Angles

Problem

While measuring the angles in a triangle, Patricia found the measure of one of the angles is $45°$. Once she had measured the other two angles, she noticed that the measures of these two angles are in the ratio $4 : 5$. What is the measure of each of the other two angles?



Solution

Solution 1

Since the measure of the unknown angles are in the ratio of $4 : 5$, then we can give the angles measures of $5n°$ and $4n°$.

Since the sum of the measures of the three angles in any triangle is $180°$, then

$$\begin{aligned} 45 + 4n + 5n &= 180 \\ 4n + 5n &= 135 \\ 9n &= 135 \\ n &= 15 \end{aligned}$$

Therefore, the other two angles are $5n = 5(15) = 75°$ and $4n = 4(15) = 60°$.

Solution 2

Since the sum of the measures of the three angles in any triangle is $180°$, then the sum of the measures of the two unknown angles in the triangle is $180° - 45° = 135°$.

The measures of the two unknown angles are in the ratio $4 : 5$, and so one of the two angles measures $\frac{5}{4+5} = \frac{5}{9}$ of the sum of the two angles, while the other angle measures $\frac{4}{4+5} = \frac{4}{9}$ of the sum of the two angles.

That is, the larger of the two unknown angles measures $\frac{5}{9} \times 135° = 75°$, and the smaller of the unknown angles measures $\frac{4}{9} \times 135° = 60°$.

Therefore, the other two angles are $75°$ and $60°$.

Problem of the Week

Problem C

Five Magnets

Harlow has five magnets, each with a different number from 1 to 5. They arranged these magnets to create a five digit number $ABCDE$ such that:

- the three-digit number ABC is divisible by 4,
- the three-digit number BCD is divisible by 5, and
- the three-digit number CDE is divisible by 3.

Determine the five-digit number that Harlow created.

Theme: Number Sense

Problem of the Week
Problem C and Solution
Five Magnets

Problem

Harlow has five magnets, each with a different number from 1 to 5. They arranged these magnets to create a five digit number $ABCDE$ such that:

- the three-digit number ABC is divisible by 4,
- the three-digit number BCD is divisible by 5, and
- the three-digit number CDE is divisible by 3.

Determine the five-digit number that Harlow created.

Solution

Since ABC is divisible by 4, it follows that C must be even, so $C = 2$ or $C = 4$.

Since BCD is divisible by 5, it follows that $D = 0$ or $D = 5$. However, there is no magnet with a 0, so it follows that $D = 5$.

We also know that CDE is divisible by 3. We can consider the following two cases.

- **Case 1:** $C = 2$
  If $C = 2$, then the three-digit number CDE is $25E$. The only possibilities for E are 1, 3, or 4. However, none of 251, 253 and 254 are divisible by 3. It follows that C cannot equal 2.
- **Case 2:** $C = 4$
  If $C = 4$ then the three-digit number CDE is $45E$. The only possibilities for E are 1, 2, or 3. Since 451 and 452 are not divisible by 3, but 453 is divisible by 3, it follows that $C = 4$ and $E = 3$.

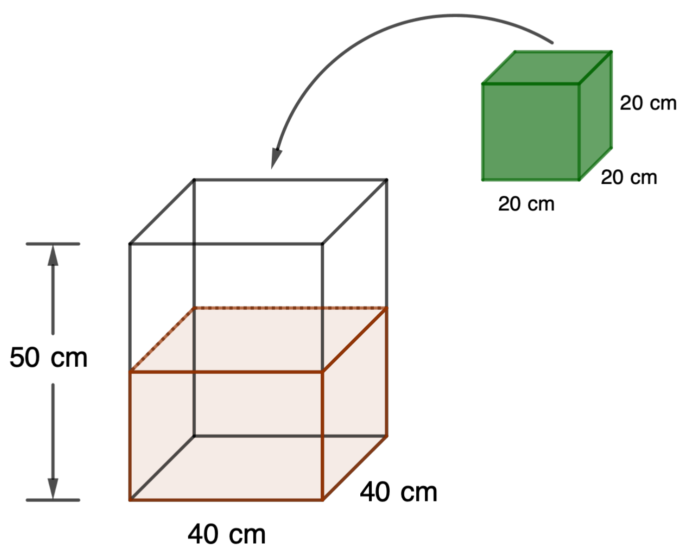
Thus, the three-digit number ABC is $AB4$. The only magnets not used yet are numbered 1 and 2, so this number is 124 or 214. Since 214 is not divisible by 4, but 124 is divisible by 4, it follows that $A = 1$ and $B = 2$.

Therefore, the five-digit number must be 12453.

Problem of the Week
Problem C
Reach for the Top

Priya has a container in the shape of a rectangular prism with base 40 cm by 40 cm and height 50 cm. She fills the container with water so that the water reaches half of the height of the container. A solid cube with side length 20 cm is then placed in the container.

How far from the top of the container does the water now reach?



Themes: Geometry & Measurement, Number Sense

Problem of the Week

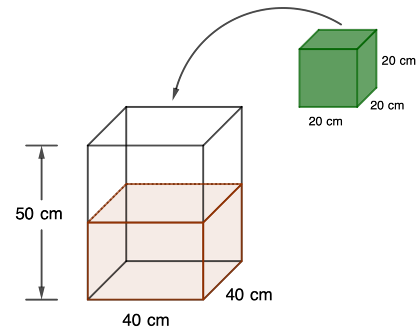
Problem C and Solution

Reach for the Top

Problem

Priya has a container in the shape of a rectangular prism with base 40 cm by 40 cm and height 50 cm. She fills the container with water so that the water reaches half of the height of the container. A solid cube with side length 20 cm is then placed in the container.

How far from the top of the container does the water now reach?



Solution

Solution 1

The height of the water is $\frac{50}{2} = 25$ cm. Since volume = length × width × height, the volume of water in the container is $40 \times 40 \times 25 = 40\,000$ cm^3.

The volume of the solid cube is $20 \times 20 \times 20 = 8000$ cm^3.

Thus, the total volume is $40\,000 + 8000 = 48\,000$ cm^3.

Let x represent the height of the water, in centimetres, after the cube is added. Then, using the formula for volume of a rectangular solid with 40 cm by 40 cm base, we have

$$\begin{aligned} 48\,000 &= 40 \times 40 \times x \\ 48\,000 &= 1600 \times x \\ x &= 30 \end{aligned}$$

Therefore, the new water height is 30 cm and the water is $50 - 30 = 20$ cm from the top of the container.

Solution 2

Let h be the height of a rectangular prism with base 40 cm by 40 cm and with the same volume as the solid cube. Since volume = length × width × height, we have

$$\begin{aligned} 40 \times 40 \times h &= 20 \times 20 \times 20 \\ 1600 \times h &= 8000 \\ h &= 5 \end{aligned}$$

Therefore, increasing the height of water in the container by 5 cm will increase the volume by 8000 cm^3, which is equal to the volume of the solid cube.

Therefore, the new water height is $25 + 5 = 30$ cm and the water is $50 - 30 = 20$ cm from the top of the container.

Problem of the Week
Problem C
All the Digits

Georgina listed the integers from 1 to 13:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$$

She determined that the sum of all of the digits of the integers in this list is

$$1+2+3+4+5+6+7+8+9+(1+0)+(1+1)+(1+2)+(1+3)=55$$

She then challenged you to determine the sum of all of the digits of the integers from 1 to 100. What sum would you get?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Theme: Number Sense

Problem of the Week
Problem C and Solution
All the Digits

Problem

Georgina listed the integers from 1 to 13:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$$

She determined that the sum of all of the digits of the integers in this list is
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + (1 + 0) + (1 + 1) + (1 + 2) + (1 + 3) = 55$$

She then challenged you to determine the sum of all of the digits of the integers from 1 to 100. What sum would you get?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Solution

In the integers from 1 to 100, each digit from 0 to 9 appears as the units digit of a number exactly ten times. For example, the digit 1 appears as the units digit in the numbers 1, 11, 21, 31, 41, 51, 61, 71, 81, 91, a total of 10 numbers. Therefore, the sum of all of the units digits is

$$\begin{aligned} &10(1) + 10(2) + 10(3) + 10(4) + 10(5) + 10(6) + 10(7) + 10(8) + 10(9) + 10(0)\\ &\quad = 10(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 0)\\ &\quad = 10(45)\\ &\quad = 450 \end{aligned}$$

Similarly, each digit from 1 to 9 appears as the tens digit of a number exactly ten times. For example, the digit 1 appears as the tens digit in the numbers 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, a total of 10 numbers. The digit 0 appears as the tens digit of a number once. Therefore, the sum of all of the tens digits is

$$\begin{aligned} 10(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) + 0 &= 10(45)\\ &= 450 \end{aligned}$$

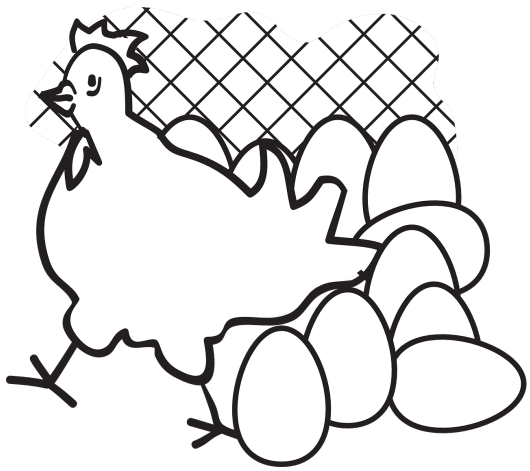
The number 100 is the only number with a hundreds digit. We need to add 1 to our final sum.

Therefore, the sum of all of the digits of the integers from 1 to 100 is $450 + 450 + 1 = 901$.

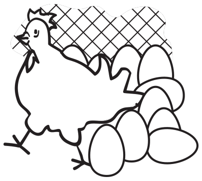
Problem of the Week
Problem C
An Average Error

At POTW Farms, Maggie is responsible for collecting eggs each day. She recorded the number of eggs collected each day for one week, and then calculated that the average number of eggs collected in one day that week was 77. When she double-checked the number of eggs she recorded for each day, she noticed that she had recorded 18 eggs on Wednesday. However, she had actually collected 81 eggs.

What is the corrected average number of eggs collected in one day for that week?



Theme: Number Sense



Problem of the Week
Problem C and Solution
An Average Error

Problem

At POTW Farms, Maggie is responsible for collecting eggs each day. She recorded the number of eggs collected each day for one week, and then calculated that the average number of eggs collected in one day that week was 77. When she double-checked the number of eggs she recorded for each day, she noticed that she had recorded 18 eggs on Wednesday. However, she had actually collected 81 eggs.

What is the corrected average number of eggs collected in one day for that week?

Solution

Solution 1

This first solution works primarily with the definition of an average.

To calculate the average, we add the recorded number of eggs for each of the seven days and divide by 7.

$$\frac{\text{sum of recorded number eggs for the seven days}}{7} = 77$$

To then obtain the sum of the recorded number of eggs collected for the seven days, we would multiply the average by 7.

$$\text{sum of recorded number of eggs for the seven days} = 7 \times 77 = 539$$

Now this sum includes the wrong number of eggs recorded on Wednesday of 18. So, we need to adjust the sum by subtracting the wrong number of eggs and adding the corrected number of eggs.

$$\text{correct sum of number of eggs for the seven days} = 539 - 18 + 81 = 602$$

We can now obtain Maggie's corrected average by dividing the corrected total by 7.

$$\text{correct average} = \frac{\text{correct sum of number of eggs for the seven days}}{7} = \frac{602}{7} = 86$$

Therefore, the corrected average is 86 eggs per day.

Solution 2

The second solution looks at how an increase in the number of eggs on one day will affect an overall average.

For an average based on seven days, an increase of 1 egg will cause the overall average to increase by $\frac{1}{7}$ of 1 egg. So, for each increase of 7 eggs, the overall average will increase by 1 egg. That is, the average number eggs will increase by 1 egg for every 7 eggs increased on Wednesday.

The number of eggs on Wednesday increases by $81 - 18 = 63$ eggs. Since $63 \div 7 = 9$, her average will increase by 9 eggs from 77 to 86 eggs.

Therefore, the corrected average is 86 eggs per day.

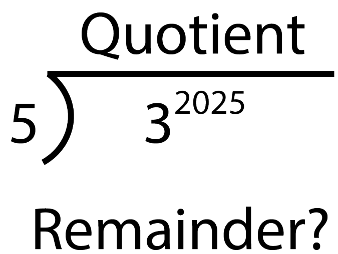
Problem of the Week

Problem C

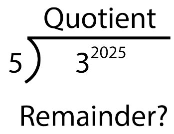
What is the Remainder?

3^6 means $3 \times 3 \times 3 \times 3 \times 3 \times 3$ and is equal to 729 when expressed as an integer. When 3^6 is divided by 5, the remainder is 4.

When the integer 3^{2025} is divided by 5, what is the remainder?



Theme: Number Sense



Problem of the Week
Problem C and Solution
What is the Remainder?

Problem

3^6 means $3 \times 3 \times 3 \times 3 \times 3 \times 3$ and is equal to 729 when expressed as an integer. When 3^6 is divided by 5, the remainder is 4.

When the integer 3^{2025} is divided by 5, what is the remainder?

Solution

When the units digit of an integer is 0 or 5, the remainder is 0 when divided by 5. When the units digit of an integer is 1 or 6, the remainder is 1 when divided by 5. For example, $56 \div 5 = 11$ R1. When the units digit of an integer is 2 or 7, the remainder is 2 when divided by 5. When the units digit of an integer is 3 or 8, the remainder is 3 when divided by 5. When the units digit of an integer is 4 or 9, the remainder is 4 when divided by 5.

Let's examine the pattern of the units digits of the first eight powers of 3.

Power	Value	Units Digit
3^1	3	3
3^2	9	9
3^3	27	7
3^4	81	1
3^5	243	3
3^6	729	9
3^7	2187	7
3^8	6561	1

The units digits appears to repeat every four powers of 3. To convince ourselves that this pattern continues, we notice that when starting with the integer 3 and continually multiplying by 3, there are a limited number of options for the resulting product's units digit. For any integer whose units digit is 3, when multiplied by 3, the resulting product's units digit is 9. For any integer whose units digit is 9, when multiplied by 3, the resulting product's units digit is 7. For any integer whose units digit is 7, when it is multiplied by 3, the resulting product's units digit is 1. For any integer whose units digit is 1, when it is multiplied by 3, the resulting product's units digit is 3. Thus, the cycle repeats.

Another way to convince ourselves that the pattern continues is by considering the fact that any positive integer can be written as $10a + b$, where a and b are integers and $a \geq 0$ and $0 \leq b \leq 9$. Its product with 3 can be written as $3(10a + b) = 3(10a) + 3b$. Since $3(10a) = 3a(10)$, it must have a units digit of 0. Therefore, the units digit of $3(10a + b)$ must be the same as the units digit of $3b$. Thus, for any integer with units digit is 3 (so $b = 3$), when multiplied by 3, the resulting product's units digit is 9. For any integer whose units digit is 9 (so $b = 9$), when multiplied by 3, the resulting product's units digit is 7. For any integer whose units digit is 7 (so $b = 7$), when multiplied by 3, the resulting product's units digit is 1. For any integer whose units digit is 1 (so $b = 1$), when multiplied by 3, the resulting product's units digit is 3. Since the first power of 3 in our list, 3^1, has $b = 3$, we get the repeating cycle 3, 9, 7, and 1 of units digits.

Thus, we need to determine how many cycles of four there are in 2025 and if any remainder exists. We determine that $2025 \div 4 = 506$ R1. The remainder 1 corresponds to the first term of the cycle whose units digit is 3, meaning that the units digit of 3^{2025} is 3. Since the units digit is 3, the remainder when 3^{2025} is divided by 5 is 3.

Therefore, when 3^{2025} is divided by 5, the remainder is 3.

Problem of the Week

Problem C

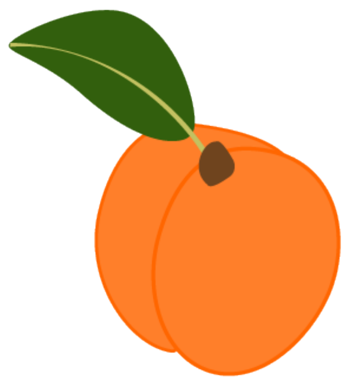
Drying Fruit

Fruit can be preserved through drying to remove excess moisture.

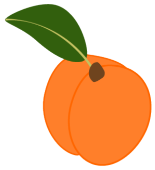
The water content of a certain fruit, by mass, is 70%. Therefore, 30% of the fruit, by mass, is other material.

When left in the sun to dry, the fruit loses 80% of its water content, and the amount of other material remains the same.

Rounded to the nearest tenth, what percent of the dried fruit is water?



Theme: Number Sense



Problem of the Week
Problem C and Solution
Drying Fruit

Problem

Fruit can be preserved through drying to remove excess moisture.

The water content of a certain fruit, by mass, is 70%. Therefore, 30% of the fruit, by mass, is other material.

When left in the sun to dry, the fruit loses 80% of its water content, and the amount of other material remains the same.

Rounded to the nearest tenth, what percent of the dried fruit is water?

Solution

Solution 1

Let's consider a piece of fruit that originally weighs 100 g. Since 70% of the mass is water, that means that 70 g is water and 30 g is other material.

When left in the sun to dry, the fruit loses 80% of its water mass. So it loses 80% of 70 g $= 0.8 \times 70 = 56$ g of water, and $70 - 56 = 14$ g of water remains.

The dried fruit still contains 30 g of other material. Therefore, the dried fruit consists of 14 g of water and 30 g other material, for a total of 44 g.

Therefore, the dried fruit is $\dfrac{14}{14+30} \times 100\% = \dfrac{14}{44} \times 100\% \approx 31.8\%$ water.

Solution 2

Suppose the fruit originally weighs x g. Since 70% of the mass is water, that means that 70% of $x = 0.7 \times x = 0.7x$ g is water and 30% of $x = 0.3x$ g is other material.

When left in the sun to dry, the fruit loses 80% of its water mass. So it loses 80% of $0.7x = 0.8 \times 0.7x = 0.56x$ g of water, and therefore $0.7x - 0.56x = 0.14x$ g of water remains.

The dried fruit still contains $0.3x$ g of other material. Therefore, the dried fruit consists of $0.14x$ g of water and $0.3x$ g of other material, for a total of $0.44x$ g.

Therefore, the dried fruit is $\dfrac{0.14x}{0.14x + 0.30x} \times 100\% = \dfrac{0.14x}{0.44x} \times 100\% \approx 31.8\%$ water.

Problem of the Week

Problem C

The Smallest Part

Suppose $N = 2^2 \times 3^2 \times 5^2 \times k$, where k is a positive integer. If N is divisible by 2025, then what is the smallest possible value for k?

$$N = 2^2 \times 3^2 \times 5^2 \times k$$

Theme: Number Sense

Problem of the Week
Problem C and Solution
The Smallest Part

Problem

Suppose $N = 2^2 \times 3^2 \times 5^2 \times k$, where k is a positive integer. If N is divisible by 2025, then what is the smallest possible value for k?

$$N = 2^2 \times 3^2 \times 5^2 \times k$$

Solution

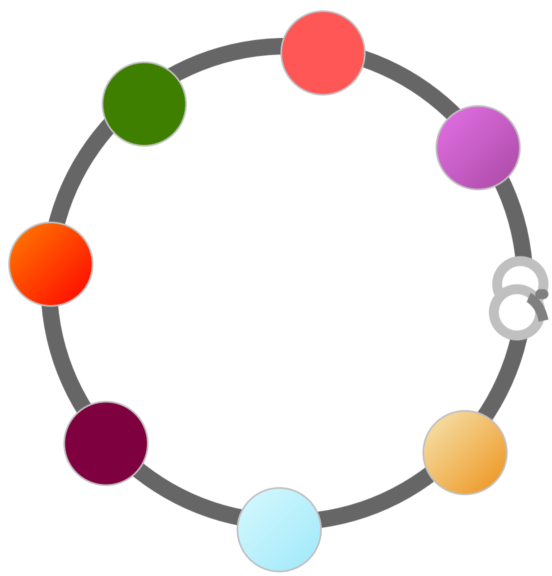
First we note that $2025 = 3^4 \times 5^2$. Then, since N is divisible by 2025, N must have at least four factors of 3 and at least two factors of 5.

N already has two factors of 3 and two factors of 5. Thus, N needs at least two more factors of 3 in order to make it divisible by 2025. Therefore, the smallest possible value for k is $3^2 = 9$.

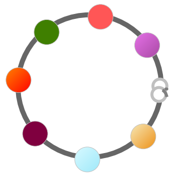
Problem of the Week
Problem C
Will it be Four or Seven?

Sophia has been making two types of necklaces: small necklaces that contain four beads each and large necklaces that contain seven beads each.

After creating a certain number of small and large necklaces, a total of 99 beads have been used. Determine all possibilities for how many of each type of necklace Sophia has made.



Theme: Number Sense



Problem of the Week
Problem C and Solution
Will it be Four or Seven?

Problem

Sophia has been making two types of necklaces: small necklaces that contain four beads each and large necklaces that contain seven beads each.

After creating a certain number of small and large necklaces, a total of 99 beads have been used. Determine all possibilities for how many of each type of necklace Sophia has made.

Solution

Let S represent the number of small necklaces and L represent the number of large necklaces she has made. Since S and L represent numbers of necklaces, both must be integers greater than or equal to 0. Since the small necklaces use 4 beads each, S necklaces would use $4 \times S$ or $4S$ beads in total. Since the large necklaces use 7 beads each, L necklaces would use $7 \times L$ or $7L$ beads in total. Since a total of 99 beads have been used, $4S + 7L = 99$.

We can also determine a maximum value for L. Since each large necklace uses 7 beads, $99 \div 7 \approx 14.1$, and L must be an integer, we know that L must be less than or equal to 14. Thus, L is an integer greater than or equal to 0 and less than or equal to 14. We could at this point check all of the possible integer values for L from 0 to 14. However, we can narrow down the possibilities even more.

In the equation, $4S + 7L = 99$, $4S$ will always be an even integer since 4 times any integer is always even. We have the even integer $4S$ plus $7L$ is equal to the odd integer 99. This means that $7L$ must be an odd integer. (The sum of an even integer and an even integer is an even integer, not an odd integer.) For $7L$ to be an odd integer, L must be odd. (If L is even, $7L$ would be even.) This observation reduces the possible values for L to the odd positive integers between 0 and 14, namely 1, 3, 5, 7, 9, 11, 13. For each possible value of L, we now determine $7L$, the total number of large beads used, $4S = 99 - 7L$, the total number of small beads used, and finally $S = (99 - 7L) \div 4$, the number of small necklaces for that value of L. If S is a non-negative integer, then we have found a valid possibility.

L	$7L$	$4S = 99 - 7L$	$S = (99-7L) \div 4$	Valid Possibility?
1	$7 \times 1 = 7$	$99 - 7 = 92$	$92 \div 4 = 23$	Yes, S is an integer
3	$7 \times 3 = 21$	$99 - 21 = 78$	$78 \div 4 = 19.5$	No, S is not an integer
5	$7 \times 5 = 35$	$99 - 35 = 64$	$64 \div 4 = 16$	Yes, S is an integer
7	$7 \times 7 = 49$	$99 - 49 = 50$	$50 \div 4 = 12.5$	No, S is not an integer
9	$7 \times 9 = 63$	$99 - 63 = 36$	$36 \div 4 = 9$	Yes, S is an integer
11	$7 \times 11 = 77$	$99 - 77 = 22$	$22 \div 4 = 5.5$	No, S is not an integer
13	$7 \times 13 = 91$	$99 - 91 = 8$	$8 \div 4 = 2$	Yes, S is an integer

Therefore, Sophia has either made 1 large necklace and 23 small necklaces, or 5 large necklaces and 16 small necklaces, or 9 large necklaces and 9 small necklaces, or 13 large necklaces and 2 small necklaces.

Problem of the Week
Problem C
Hurry Up

Ethan is riding his e-scooter 6 km to basketball practice. When he left his house he checked his watch and determined that he had 24 minutes to get to practice. He started off at a speed of 12 km/hr, but after 12 min he realized he needs to increase his speed in order to get to practice on time. For the remaining time, what speed should Ethan ride at in order for the entire trip to take exactly 24 minutes?

Theme: Number Sense

Problem of the Week
Problem C and Solution
Hurry Up

Problem

Ethan is riding his e-scooter 6 km to basketball practice. When he left his house he checked his watch and determined that he had 24 minutes to get to practice. He started off at a speed of 12 km/hr, but after 12 min he realized he needs to increase his speed in order to get to practice on time. For the remaining time, what speed should Ethan ride at in order for the entire trip to take exactly 24 minutes?

Solution

The entire trip is 6 km and should take 24 minutes. Already Ethan has traveled for 12 min. That means the rest of his trip should take $24 - 12 = 12$ minutes.

For the first 12 minutes, Ethan rode at 12 km/h. This means that at that speed, in one hour, he would have traveled 12 km. Since 12 minutes is equal to $\frac{12}{60} = \frac{1}{5}$ of an hour, then the distance traveled during the first 12 minutes is $\frac{1}{5}$ of 12 km. This is $12 \div 5 = 2.4$ km. Since the total distance is 6 km, Ethan has to travel $6 - 2.4 = 3.6$ km in the remaining 12 min.

We need to determine the speed that Ethan should ride at to accomplish this. Ethan needs to travel 3.6 km in 12 min, or $\frac{1}{5}$ of an hour. At this speed, Ethan would travel $3.6 \times 5 = 18$ km in one hour. Therefore, Ethan must ride at 18 km/hr for the remaining time in order for the trip to take exactly 24 minutes.

Problem of the Week

Problem C

1000 is SUM Number!

Did you know that 1000 can be written as the sum of 16 consecutive integers?

That is,

$$1000 = 55+56+57+58+59+60+61+62+63+64+65+66+67+68+69+70$$

The notation below illustrates a mathematical short form used for writing the above sum. The notation is called *Sigma Notation*.

$$\sum_{i=55}^{70} i = 1000$$

Using at least two integers, what is the minimum number of consecutive integers that sum to exactly 1000?

Themes: Algebra, Number Sense

$$\sum_{i=55}^{70} i = 1000$$

Problem of the Week

Problem C and Solution

1000 is SUM Number!

Problem

Did you know that 1000 can be written as the sum of 16 consecutive integers?

That is,

$$1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70$$

Using at least two integers, what is the minimum number of consecutive integers that sum to exactly 1000?

Solution

We will start with using two integers and then increase the number of integers by 1, until we discover the first number of integers that works.

- Can 1000 be written as the sum of two consecutive integers?

  Let n and $n + 1$ represent the two integers. Then we have

$$\begin{aligned} n + (n + 1) &= 1000 \\ 2n + 1 &= 1000 \\ 2n &= 999 \\ n &= 499.5 \end{aligned}$$

  Since n is not an integer, it is not possible to write 1000 using two consecutive integers.

- Can 1000 be written as the sum of three consecutive integers?

  Let n, $n + 1$, $n + 2$ represent the three integers. Then we have

$$\begin{aligned} n + (n + 1) + (n + 2) &= 1000 \\ 3n + 3 &= 1000 \\ 3n &= 997 \\ n &\approx 332.3 \end{aligned}$$

  Since n is not integer, it is not possible to write 1000 using three consecutive integers. (Refer to the note following the solution for an alternate way to represent three consecutive integers.)

- Can 1000 be written as the sum of four consecutive integers?

  Let n, $n + 1$, $n + 2$, $n + 3$ represent the four integers. Then we have

$$\begin{aligned} n + (n + 1) + (n + 2) + (n + 3) &= 1000 \\ 4n + 6 &= 1000 \\ 4n &= 994 \\ n &= 248.5 \end{aligned}$$

  Since n is not an integer, it is not possible to write 1000 using four consecutive integers.

- Can 1000 be written as the sum of five consecutive integers?

  Let n, $n+1$, $n+2$, $n+3$, $n+4$ represent the five integers. Then we have

$$\begin{aligned} n+(n+1)+(n+2)+(n+3)+(n+4) &= 1000 \\ 5n+10 &= 1000 \\ 5n &= 990 \\ n &= 198 \end{aligned}$$

  Since n is an integer, it is possible to write 1000 using five consecutive integers. That is, as $1000 = 198 + 199 + 200 + 201 + 202$.

Therefore, the minimum number of consecutive integers that sum to 1000 is five.

Note:

In the second case, when we checked if 1000 could be written as the sum of three consecutive integers, we could have proceeded as follows:

Let $a-1$, a, $a+1$ represent the three consecutive integers. Then we have

$$\begin{aligned} (a-1)+a+(a+1) &= 1000 \\ 3a &= 1000 \\ a &\approx 333.3 \end{aligned}$$

Since a is not an integer, it is not possible to write 1000 using three consecutive integers.

This idea is useful when we are finding the sum of an odd number of consecutive integers. If we applied the same idea to the fourth case by using $a-2$, $a-1$, a, $a+1$, $a+2$ to represent the five consecutive integers, we would have

$$\begin{aligned} (a-2)+(a-1)+a+(a+1)+(a+2) &= 1000 \\ 5a &= 1000 \\ a &= 200 \end{aligned}$$

Thus, $a-2=198$, $a-1=199$, $a+1=201$, and $a+2=202$. And so it is possible to write 1000 using five consecutive integers as $1000 = 198 + 199 + 200 + 201 + 202$.

For Further Thought:

What is the largest **odd** number of consecutive positive integers that can be used to sum to 1000?

How would your answer change if the word positive was removed from the above sentence?

Problem of the Week

Problem C

Efficient Tiling

Sophia has an unlimited supply of square tiles. Sophia has 1 cm by 1 cm tiles, 2 cm by 2 cm tiles, 3 cm by 3 cm tiles, and so on. Every tile has integer side lengths, in centimetres.

A rectangular bathroom floor with an 84 cm by 140 cm surface is to be completely covered by identical square tiles, none of which can be cut. Sophia knows that the floor can be completely covered with 11 760 tiles of size 1 cm by 1 cm, since $84 \times 140 = 11\,760$ cm^2. However, Sophia wants to use the minimum number of identical sized tiles to complete the job.

Determine the minimum number of identical sized tiles required to completely cover the bathroom floor.

Themes: Geometry & Measurement, Number Sense

Problem of the Week
Problem C and Solution
Efficient Tiling

Problem

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Determine the minimum number of identical sized tiles required to completely cover the bathroom floor.

Solution

To use the least number of tiles, Sophia must use the largest tile possible. The square tile must have sides less than or equal to 84 cm. If the tile has side length greater than 84 cm, it would have to be cut to fit the width of the floor.

Since the tiles are square and must completely cover the floor, the side length of the tile must be a number that is a factor of both 84 and 140. In fact, since Sophia wants the largest side length, she is looking for the greatest common factor of 84 and 140.

The positive factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84.

The positive factors of 140 are 1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70, and 140.

The largest number common to both lists is 28. Therefore, the greatest common factor of 84 and 140 is 28. The required tiles are 28 cm by 28 cm. Since $84 \div 28 = 3$, the surface is 3 tiles wide. Since $140 \div 28 = 5$, the surface is 5 tiles long. The minimum number of tiles required is $3 \times 5 = 15$ tiles.

The number of 28 cm by 28 cm tiles required to cover the floor area is 15. This is the minimum number of tiles required.

Problem of the Week
Problem C
Group Work

Piero has 7 tiles, each with a different integer from 1 to 7 written on it. There are many ways in which he can separate his tiles into groups, where each group contains at least one tile. For example, he can separate his tiles into 3 groups as shown.

The sum of the numbers in each of these groups is 13, 10, and 5, respectively.

Piero then separates his tiles into different groups and notices that the sum of the numbers in each group is the same. In how many different ways can Piero separate his tiles into at least two groups so that the sum of the numbers in each group is the same?

Themes: Computational Thinking, Number Sense

Problem of the Week

Problem C and Solution

Group Work

Problem

Piero has 7 tiles, each with a different integer from 1 to 7 written on it. There are many ways in which he can separate his tiles into groups, where each group contains at least one tile. For example, he can separate his tiles into 3 groups as shown.

The sum of the numbers in each of these groups is 13, 10, and 5, respectively.

Piero then separates his tiles into different groups and notices that the sum of the numbers in each group is the same. In how many different ways can Piero separate his tiles into at least two groups so that the sum of the numbers in each group is the same?

Solution

The sum of the numbers on Piero's tiles is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. If the sum of the numbers in each group is the same, then this sum must be a factor of 28. The positive factors of 28 are 1, 2, 4, 7, 14, and 28. Since the number 7 must be in one of the groups, it follows that the sum of each group must be at least 7. Also since there must be at least 2 groups, then the sum of each group cannot be 28, because in that case there would be only 1 group. It follows that the groups could each have a sum of 7 or a sum of 14. We will look at these two cases.

Case 1: Each group has a sum of 7.
In this case, there would be $28 \div 7 = 4$ groups. The only way to separate the tiles into 4 groups, each with a sum of 7, is as $\{7\},\ \{1, 6\},\ \{2, 5\},\ \{3, 4\}$.

Case 2: Each group has a sum of 14.
In this case, there would be $28 \div 14 = 2$ groups. To help count the possible groups, we note that one of the two groups must contain the tile with the 7. The possibilities for the other tiles are as follows:

- If the group with the 7 also contains the 6, then the only possibility is $\{1, 6, 7\}$. The other group must then be $\{2, 3, 4, 5\}$.
- If the group with the 7 also contains the 5, then the only possibility is $\{2, 5, 7\}$. The other group must then be $\{1, 3, 4, 6\}$.

- If the group with the 7 also contains the 4, then the group could be either $\{3, 4, 7\}$ or $\{1, 2, 4, 7\}$. The other group would then be $\{1, 2, 5, 6\}$ or $\{3, 5, 6\}$, respectively.
- If the group with the 7 also contains the 3, 2, or 1, then it must also contain the 4, 5, or 6, since $7 + 3 + 2 + 1 = 13$, which is less than 14. So there are no additional cases.

Thus, there are 4 ways to separate the tiles into 2 groups, each with a sum of 14.

Therefore, there are $1 + 4 = 5$ different ways in which Piero can separate his tiles into at least two groups so that the sum of the numbers in each group is the same. These are listed below.

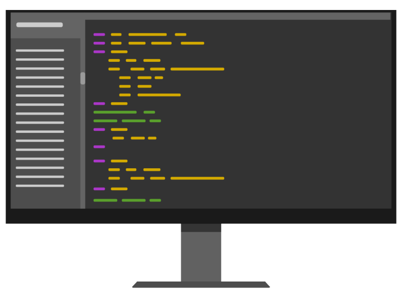
- $\{7\},\ \{1, 6\},\ \{2, 5\},\ \{3, 4\}$
- $\{1, 6, 7\},\ \{2, 3, 4, 5\}$
- $\{2, 5, 7\},\ \{1, 3, 4, 6\}$
- $\{3, 4, 7\},\ \{1, 2, 5, 6\}$
- $\{1, 2, 4, 7\},\ \{3, 5, 6\}$

Problem of the Week
Problem C
Sum Program

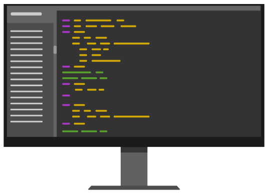
Ruben wrote a program that takes a list of numbers as input. The program then adds each pair of adjacent numbers in the list to obtain a new list of numbers, then repeats this process with the new list until the list contains only one number. This number is then the output.

For example, if the input is $(1,\ 5,\ 3,\ 2)$ then the program adds the adjacent numbers in the list to obtain $(6,\ 8,\ 5)$. The program then repeats this process to obtain $(14,\ 13)$, and then (27). Since the list now contains only one number, the output is 27.

Let R represent Ruben's favourite number. Ruben input the numbers $(6,\ 4,\ R,\ 7)$ into his program and the output was 4 times his favourite number, or $4R$. What is Ruben's favourite number?



Themes: Algebra, Number Sense



Problem of the Week
Problem C and Solution
Sum Program

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Let R represent Ruben's favourite number. Ruben input the numbers $(6,\ 4,\ R,\ 7)$ into his program and the output was 4 times his favourite number, or $4R$. What is Ruben's favourite number?

Solution

We could attempt this problem using trial and error, and perhaps we would even stumble on the correct answer. However, this may take a long time and is not an efficient method for solving the problem. Instead we will provide a more algebraic solution.

The list of numbers inputted into the program was $(6,\ 4,\ R,\ 7)$. After the adjacent numbers are added together, we obtain $(10,\ 4+R,\ R+7)$. After the adjacent numbers in this list are added together, we obtain $(10+4+R,\ 4+R+R+7)$, which is equal to $(14+R,\ 11+2R)$. After the adjacent numbers in this list are added together, we obtain $(14+R+11+2R)$ which is equal to $(25+3R)$. Since there is now one number in the list, this number should be equal to the output of the program, which is $4R$. Thus,

$$\begin{aligned} 25+3R &= 4R \\ 25+3R-3R &= 4R-3R \\ 25 &= R \end{aligned}$$

Therefore, Ruben's favourite number is 25.