Problem of the Week
Problem C and Solution
Group Work

Problem

Piero has 7 tiles, each with a different integer from 1 to 7 written on it. There are many ways in which he can separate his tiles into groups, where each group contains at least one tile. For example, he can separate his tiles into 3 groups as shown.

The sum of the numbers in each of these groups is 13, 10, and 5, respectively.

Piero then separates his tiles into different groups and notices that the sum of the numbers in each group is the same. In how many different ways can Piero separate his tiles into at least two groups so that the sum of the numbers in each group is the same?

Solution

The sum of the numbers on Piero's tiles is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. If the sum of the numbers in each group is the same, then this sum must be a factor of 28. The positive factors of 28 are 1, 2, 4, 7, 14, and 28. Since the number 7 must be in one of the groups, it follows that the sum of each group must be at least 7. Also since there must be at least 2 groups, then the sum of each group cannot be 28, because in that case there would be only 1 group. It follows that the groups could each have a sum of 7 or a sum of 14. We will look at these two cases.

Case 1: Each group has a sum of 7.
In this case, there would be $28 \div 7 = 4$ groups. The only way to separate the tiles into 4 groups, each with a sum of 7, is as $\{7\},\ \{1, 6\},\ \{2, 5\},\ \{3, 4\}$.

Case 2: Each group has a sum of 14.
In this case, there would be $28 \div 14 = 2$ groups. To help count the possible groups, we note that one of the two groups must contain the tile with the 7. The possibilities for the other tiles are as follows:

- If the group with the 7 also contains the 6, then the only possibility is $\{1, 6, 7\}$. The other group must then be $\{2, 3, 4, 5\}$.
- If the group with the 7 also contains the 5, then the only possibility is $\{2, 5, 7\}$. The other group must then be $\{1, 3, 4, 6\}$.

- If the group with the 7 also contains the 4, then the group could be either $\{3, 4, 7\}$ or $\{1, 2, 4, 7\}$. The other group would then be $\{1, 2, 5, 6\}$ or $\{3, 5, 6\}$, respectively.
- If the group with the 7 also contains the 3, 2, or 1, then it must also contain the 4, 5, or 6, since $7 + 3 + 2 + 1 = 13$, which is less than 14. So there are no additional cases.

Thus, there are 4 ways to separate the tiles into 2 groups, each with a sum of 14.

Therefore, there are $1 + 4 = 5$ different ways in which Piero can separate his tiles into at least two groups so that the sum of the numbers in each group is the same. These are listed below.

- $\{7\},\ \{1, 6\},\ \{2, 5\},\ \{3, 4\}$
- $\{1, 6, 7\},\ \{2, 3, 4, 5\}$
- $\{2, 5, 7\},\ \{1, 3, 4, 6\}$
- $\{3, 4, 7\},\ \{1, 2, 5, 6\}$
- $\{1, 2, 4, 7\},\ \{3, 5, 6\}$