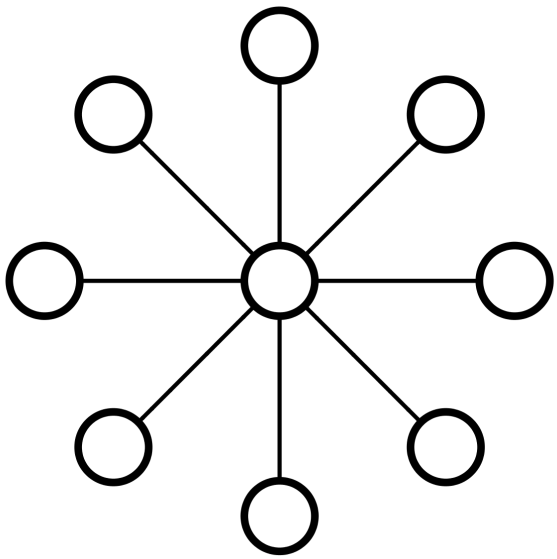
Problem of the Week
Problem C and Solution
The Snowflake Game

Problem

The Snowflake Game is played on a board that consists 9 circles. There is 1 circle in the centre of the board, with 4 line segments through it, and a circle at the end of each line segment.



Two players alternate turns placing discs numbered 1 to 9 in the circles on the board. Each number can only be used once in any game. The object of the game is to be the first player to place a disc so that the sum of the three numbers along a line segment through the centre circle is exactly 15.

Alex and Blake play the game. Alex goes first. Show that Alex has a *winning strategy* if she places a 6 in the centre circle on her first turn. That is, show that if Alex places a 6 in the centre circle on her first turn, then no matter which numbers Blake plays, Alex can always win the game.

Solution

If Alex places a 6 in the centre circle on her first turn, then the other two discs in the line segment would need to add to 9 in order for the sum to be 15.

Since each number can be used only once in a game, after Alex places the 6, the numbers that remain are 1, 2, 3, 4, 5, 7, 8, and 9.

- **Case 1:** Blake places a 1, 2, 4, 5, 7, or 8 on his first turn.

  Then there is always an unused number remaining that Alex can place on the same line segment to win the game on her second turn. That is, if Blake

places a 1, then Alex should place an 8. If Blake places a 2, then Alex should place a 7. If Blake places a 4, then Alex should place a 5. If Blake places a 5, then Alex should place a 4. If Blake places a 7, then Alex should place a 2. If Blake places an 8, then Alex should place a 1.

- **Case 2:** Blake places a 3 on his first turn.

  Then the sum of the two discs on the line segment containing the 3 will be 9. Alex cannot win on her second turn since the only way to make the sum in that line segment 15 would be for her to place another 6. No number may be used more than once so this is not possible. However, if Alex completes the line segment by placing a 9 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7, and 8. Then, as in Case 1, no matter what Blake places on his second turn, there will always be a number that Alex can place on that same line segment so that the three numbers on the line segment sum to 15.

- **Case 3:** Blake places a 9 on his first turn.

  Then the sum of the two discs on the line segment containing the 9 will be 15. Alex cannot win on her second turn since playing any other disc on that line segment will make the sum greater than 15. However, if Alex completes the line segment by placing a 3 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7, and 8. Then, as in Case 1, no matter what Blake places on his second turn, there will always be a number that Alex can place on that same line segment so that the three numbers on the line segment sum to 15.

Therefore, we have shown that if Alex places a 6 in the centre circle on her first turn, then no matter which numbers Blake plays, Alex can always win the game on either her second or third turn.

For Further Thought:

What other numbers can Alex place in the middle circle on her first turn to have a winning strategy? That is, what other numbers can Alex place in the middle circle on her first turn so that she can always win the game, no matter which numbers Blake plays?