$$\sum_{i=55}^{70} i = 1000$$

Problem of the Week
Problem C and Solution
1000 is SUM Number!

Problem

Did you know that 1000 can be written as the sum of 16 consecutive integers?

That is,

$$1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70$$

Using at least two integers, what is the minimum number of consecutive integers that sum to exactly 1000?

Solution

We will start with using two integers and then increase the number of integers by 1, until we discover the first number of integers that works.

- Can 1000 be written as the sum of two consecutive integers?

  Let n and $n + 1$ represent the two integers. Then we have

$$\begin{aligned} n + (n + 1) &= 1000 \\ 2n + 1 &= 1000 \\ 2n &= 999 \\ n &= 499.5 \end{aligned}$$

  Since n is not an integer, it is not possible to write 1000 using two consecutive integers.

- Can 1000 be written as the sum of three consecutive integers?

  Let n, $n + 1$, $n + 2$ represent the three integers. Then we have

$$\begin{aligned} n + (n + 1) + (n + 2) &= 1000 \\ 3n + 3 &= 1000 \\ 3n &= 997 \\ n &\approx 332.3 \end{aligned}$$

  Since n is not integer, it is not possible to write 1000 using three consecutive integers. (Refer to the note following the solution for an alternate way to represent three consecutive integers.)

- Can 1000 be written as the sum of four consecutive integers?

  Let n, $n + 1$, $n + 2$, $n + 3$ represent the four integers. Then we have

$$\begin{aligned} n + (n + 1) + (n + 2) + (n + 3) &= 1000 \\ 4n + 6 &= 1000 \\ 4n &= 994 \\ n &= 248.5 \end{aligned}$$

  Since n is not an integer, it is not possible to write 1000 using four consecutive integers.

- Can 1000 be written as the sum of five consecutive integers?

  Let n, $n+1$, $n+2$, $n+3$, $n+4$ represent the five integers. Then we have

$$\begin{aligned} n+(n+1)+(n+2)+(n+3)+(n+4) &= 1000 \\ 5n+10 &= 1000 \\ 5n &= 990 \\ n &= 198 \end{aligned}$$

  Since n is an integer, it is possible to write 1000 using five consecutive integers. That is, as $1000 = 198 + 199 + 200 + 201 + 202$.

Therefore, the minimum number of consecutive integers that sum to 1000 is five.

Note:

In the second case, when we checked if 1000 could be written as the sum of three consecutive integers, we could have proceeded as follows:

Let $a-1$, a, $a+1$ represent the three consecutive integers. Then we have

$$\begin{aligned} (a-1)+a+(a+1) &= 1000 \\ 3a &= 1000 \\ a &\approx 333.3 \end{aligned}$$

Since a is not an integer, it is not possible to write 1000 using three consecutive integers.

This idea is useful when we are finding the sum of an odd number of consecutive integers. If we applied the same idea to the fourth case by using $a-2$, $a-1$, a, $a+1$, $a+2$ to represent the five consecutive integers, we would have

$$\begin{aligned} (a-2)+(a-1)+a+(a+1)+(a+2) &= 1000 \\ 5a &= 1000 \\ a &= 200 \end{aligned}$$

Thus, $a-2 = 198$, $a-1 = 199$, $a+1 = 201$, and $a+2 = 202$. And so it is possible to write 1000 using five consecutive integers as $1000 = 198 + 199 + 200 + 201 + 202$.

For Further Thought:

What is the largest **odd** number of consecutive positive integers that can be used to sum to 1000?

How would your answer change if the word positive was removed from the above sentence?