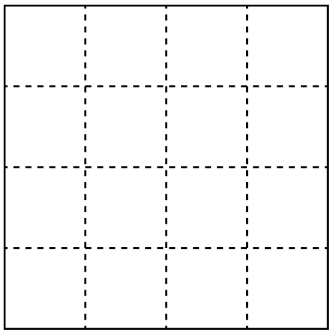
Problem of the Week
Problem B and Solution
Equal Cake and Icing

Problem

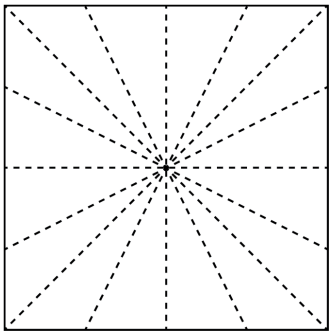
Serenity is having 16 guests for dinner. She baked a cake for dessert using a square cake pan with side length 36 cm. The cake is 8 cm tall. The top face and side faces of the cake are covered in icing.

She would like to slice the cake into 16 pieces. She calls a slicing a "fair cake" if each piece has the same amount (volume) of cake and the same amount (surface area) of icing.

(a) To cut the cake into 16 pieces, suppose she makes three equally-spaced vertical slices and three equally-spaced horizontal slices through the top face of the cake. Is this a fair cake?

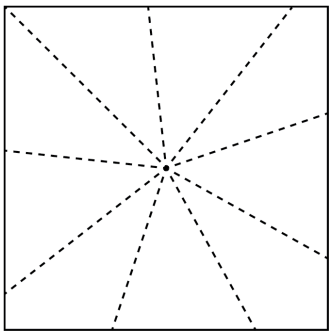


(b) To cut the cake into 16 pieces, suppose she first divides each edge of the top face into four equal lengths. She then makes a straight slice from each end of a length, through the centre of the square, to an end of a length on the opposite edge. Is this a fair cake? Show calculations to support your answer.



Extension:

Only 9 guests want to eat dessert. Serenity decides to cut the cake into 9 pieces by dividing the entire perimeter of the cake into nine equal lengths, starting in the top-left corner and moving clockwise. She then makes a slice from each end of a length to the centre of the square. Is this a fair cake? Show calculations to support your answer.



Solution

(a) Since the side length of the square pan is 36 cm, each slice has a square top face with side length $36 \div 4 = 9$ cm.

Since the height of the cake is 8 cm, the volume of each slice is $9 \times 9 \times 8 = 648$ cm^3. Thus, each slice has the same volume of cake.

The top face of each slice has $9 \times 9 = 81$ cm^2 of icing. Each side face with icing will have $9 \times 8 = 72$ cm^2 of icing. Thus, the corner pieces will have $81 + 72 + 72 = 225$ cm^2 of icing, the edge pieces that are not corner pieces will have $81 + 72 = 153$ cm^2 of icing, and the middle pieces will have only 81 cm^2 of icing.

Since each slice does not have the same amount of icing, this is not a fair cake.

(b) The top face of each slice is in the shape of a triangle. Since the side length of the square pan is 36 cm, the base of each triangle is $36 \div 4 = 9$ cm. The height of each triangle is half of the side length of the square pan, or $36 \div 2 = 18$ cm. Using the formula for area of a triangle, we have that the area of the top face of each slice is base $\times$ height $\div 2 = 9 \times 18 \div 2 = 81$ cm^2.

Since each slice has the same top face area of 81 cm^2 and same height of 8 cm, each slice has the same volume of $81 \times 8 = 648$ cm^3.

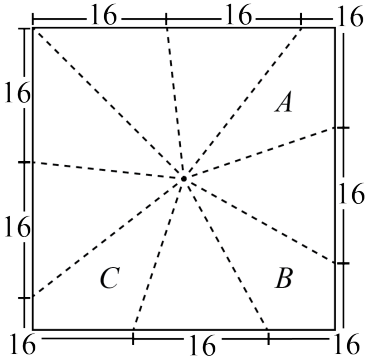
The top face of each slice has 81 cm^2 of icing. Since the base of each triangular top face is 9 cm and the height of each slice is 8 cm, each slice has a side face with $9 \times 8 = 72$ cm^2 of icing. Thus, each slice has the same amount of icing, $81 + 72 = 153$ cm^2.

Since each slice has the same volume and the same area of icing, this is a fair cake.

Solution to extension:

Since the perimeter of the cake is $36 \times 4 = 144$ cm and 9 slices are made, then each piece will have a total edge length of $\frac{144}{9} = 16$ cm with icing. Thus, since the height of the cake is 8 cm, the amount of icing on the side of each slice is $16 \times 8 = 128$ cm^2.

For six of the slices, the top face of the slice is a triangle. For the remaining three slices, the top face is a quadrilateral. These slices are marked A, B, and C.



First we look at the triangular slices. The base of each triangle is 16 cm and the height is half the side length of the square pan, or $36 \div 2 = 18$ cm. Using the formula for area of a triangle, we have that the area of the top face of each triangular slice of cake is base $\times$ height $\div 2 = 16 \times 18 \div 2 = 144$ cm^2.

Next we look at the quadrilaterals. Each quadrilateral consists of two triangles, each with height 18 cm.

The top face of slice A has one triangle with base length $36 - 16 - 16 = 4$ cm. Thus, the other triangle has base length equal to $16 - 4 = 12$ cm. Therefore, using the formula for the area of a triangle, we can determine that the area of the top face of slice A is $4 \times 18 \div 2 + 12 \times 18 \div 2 = 36 + 108 = 144$ cm^2.

The top face of slice B has one triangle with base length $36 - 12 - 16 = 8$ cm. Thus, the other triangle has base length equal to $16 - 8 = 8$ cm. Therefore, using the formula for the area of a triangle, we can determine that the area of the top face of slice B is $8 \times 18 \div 2 + 8 \times 18 \div 2 = 72 + 72 = 144$ cm^2.

The top face of slice C has one triangle with base length $36 - 8 - 16 = 12$ cm. Thus, the other triangle has base length equal to $16 - 12 = 4$ cm. Since these are equal to the base lengths of the triangles in the top face of slice A, it follows that the area of the top face of slice C is also 144 cm^2.

Therefore, since each slice has the same top face area of 144 cm^2 and the same height of 8 cm, each slice has volume equal to $144 \times 8 = 1152$ cm^2.

Also, each slice has 144 cm^2 of icing on top and 128 cm^2 of icing on the side, for a total of $144 + 128 = 272$ cm^2 of icing.

Since each slice has the same volume and the same area of icing, this is a fair cake.