Problem of the Week

Problem B and Solution

Ask the Banker

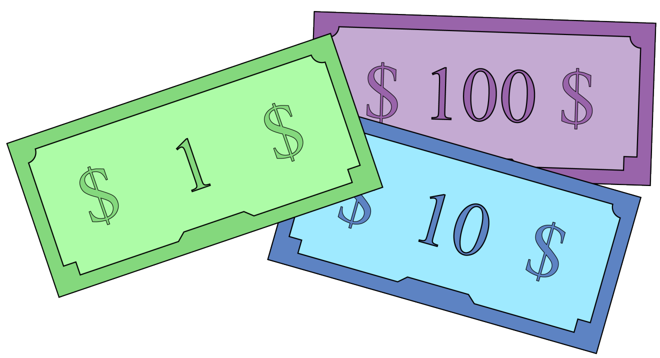
Problem

Vesna is the banker in a board game that uses \$1, \$10, and \$100 bills.

(a) Vesna needs to give a player \$2163. How can you do this using the fewest total number of bills? How can you do this using the greatest total number of bills?

(b) Is it possible to give a player \$254 using exactly 20 bills in total? How about using exactly 30 bills in total? If so, show how it's possible. If not, explain why it's not possible.

Extension: Vesna likes when she can give a player the same number of each type of bill. For which total amounts of money is this possible? Explain.



Solution

(a) To give a player \$2163 using the fewest total number of bills, we first use as many \$100 bills as we can. Since the number 2163 has 21 hundreds, then we can use at most 21 of the \$100 bills. This gives $21 \times \$100 = \$2100$, so we are left with $\$2163 - \$2100 = \$63$. Next we use as many \$10 bills as we can. Since the number 63 has 6 tens, then we can use at most 6 of the \$10 bills. This gives $6 \times \$10 = \$60$. We are then left with \$3, so we need 3 of the \$1 bills. Thus in total, we use:

$$21 \times \$100 \text{ bills}; \quad 6 \times \$10 \text{ bills}; \quad 3 \times \$1 \text{ bills}$$

This is a total of $21 + 6 + 3 = 30$ bills.

To give a player \$2163 using the greatest total number of bills, we want to use as many \$1 bills as possible. If we use all \$1 bills, then we will use 2163 bills in total.

(b) Using the strategy from (a) to use the fewest total number of bills, we can give \$254 as follows:

$$2 \times \$100 \text{ bills}; \quad 5 \times \$10 \text{ bills}; \quad 4 \times \$1 \text{ bills}$$

This uses a total of $2 + 5 + 4 = 11$ bills. Let's try replacing one \$100 bill with ten \$10 bills. This gives:

$$1 \times \$100 \text{ bill}; \quad 15 \times \$10 \text{ bills}; \quad 4 \times \$1 \text{ bills}$$

This uses a total of $1 + 15 + 4 = 20$ bills, so it is possible to give a player \$254 using exactly 20 bills in total.

Notice that every time we replace a bill with ten smaller bills, the total number of bills increases by 9. This is true if we replace one \$100 bill with ten \$10 bills or if we replace one \$10 bill with ten \$1 bills. So the total number of bills is a sequence that starts at 11 and increases by 9 each time, until it reaches 254 (which is the greatest total number of bills that can be used for \$254). Writing out more terms in the sequence gives 11, 20, 29, $38, \ldots$. Since 30 is not in this sequence, we can *not* give a player \$254 using exactly 30 bills in total.

Solution to Extension: If a player gets 1 of each bill, then the total amount is $1 \times \$100 + 1 \times \$10 + 1 \times \$1 = \$111$. If a player gets 2 of each bill, then the total amount will be $2 \times \$111 = \$222$, because they have twice as many of each bill, so the total amount will double. Similarly, if a player gets 6 of each bill, then the total amount will be $6 \times \$111 = \$666$, and if a player gets 15 of each bill, then the total amount will be $15 \times \$111 = \$1665$. Thus, the total amount will always be a multiple of 111.

We can also use variables to explain this. Suppose a player receives n bills of each type. Then the total amount is equal to $(n \times 100 + n \times 10 + n \times 1)$. This is the same as $n \times (100 + 10 + 1)$, which equals $n \times 111$. Therefore, the total amount must be a multiple of 111.