Problem of the Week

Problem A and Solution

Circular Calculations

Problem

Lorna uses the following instructions to write sequences of numbers.

Step 1: Start with a whole number greater than 0.

Step 2: If the number is even, divide it by 2 to get the next number.
If the number is odd, multiply it by 3 and add 1 to get the next number.

Step 3: Repeat Step 2 to continue the sequence.

For example, suppose Lorna starts with 9.
Since this number is odd, the next number is $9 \times 3 + 1 = 27 + 1 = 28$.
Since this number is even, the next number is $28 \div 2 = 14$.
Since this number is even, the next number is $14 \div 2 = 7$.
Thus, the first four numbers in the sequence are 9, 28, 14, and 7.

(a) Follow the instructions using the given starting number and write the first 12 numbers in each sequence.
  (i) 3 (ii) 13

(b) What do you notice about each sequence in part (a)? What would happen if you continued each sequence?

(c) Use your answer to part (b) to predict the 20^{th} number in the sequence starting with 13.

Solution

(a) The first 12 numbers in each sequence are as follows.

  (i) 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4

  (ii) 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2

(b) After each sequence reaches 1, the numbers 4, 2, and 1 repeat over and over.

(c) We can write the first 20 numbers in the sequence starting with 13 by first using part (b) to write out the first 12 numbers in the sequence, and then repeating 4, 2, and 1 until we have 20 numbers.

$$13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ 4$$

Thus, the 20^{th} number is 4.

Teacher's Notes

As an extra challenge, students could attempt this starting with the number 27. Although it takes a long time when you start with the number 27, the sequence will eventually reach 1. It takes 111 numbers to reach 1, as shown.

$27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1$

Mathematicians believe that starting with any positive integer, the instructions will always lead to a sequence that reaches 1 (or *converges* to 1). This is known as the *Collatz Conjecture*. However, proving this is true for all positive integers is an open problem. There is experimental evidence that shows this is true for very large numbers; however, there is no formal proof that the conjecture holds for all positive integers.