Problem of the Month

Solution to Problem 8: Some Surprising Squares

May 2025

1. The pair $m = 3$, $e = 1$, satisfies $m^2 - 8e^2 = 1$. We can use this solution to find the pairs m and e that satisfy $m^2 - 8e^2 = 4$ as follows:

$$4 = 4(3^2 - 8(1^2)) = (2 \cdot 3)^2 - 8(2 \cdot 1)^2 = 6^2 - 8(2)^2.$$

Similarly we have

$$9 = 9(3^2 - 8(1^2)) = (3 \cdot 3)^2 - 8(3 \cdot 1)^2 = 9^2 - 8(3)^2.$$

So, the pairs $(m, e) = (6, 2)$ and $(m, e) = (9, 3)$ satisfy $m^2 - 8e^2 = 4$ and $m^2 - 8e^2 = 9$ respectively.

2. The pair $m = 9$ and $e = 2$ satisfies $m^2 - 8e^2 = 49$. There are lots of other pairs too. Here are a few others: $(m, e) = (11, 3), (43, 15), (57, 20), (249, 88)$.

It is tempting to just imitate what we did in Question 1. However, that yields the pair $(m, e) = (21, 7)$, and 21 and 7 are *not* coprime!

3. We have

$$\begin{aligned} N((a + b\sqrt{8})(c + d\sqrt{8})) &= N(ac + 8bd + (ad + bc)\sqrt{8}) \\ &= (ac + 8bd + (ad + bc)\sqrt{8})(ac + 8bd - (ad + bc)\sqrt{8}) \\ &= (ac + 8bd)^2 - 8(ad + bc)^2 \\ &= a^2c^2 + 16abcd + 64b^2d^2 - 8a^2d^2 - 16abcd - 8b^2c^2 \\ &= (a^2 - 8b^2)(c^2 - 8d^2) \\ &= (a + b\sqrt{8})(a - b\sqrt{8})(c + d\sqrt{8})(c - d\sqrt{8}) \\ &= N(a + b\sqrt{8})N(c + d\sqrt{8}). \end{aligned}$$

4. There are two important observations to make here. First, $391 = 17 \times 23$. Second, is that $N(a + b\sqrt{8}) = a^2 - 8b^2$. So, finding integers a, b satisfying $a^2 - 8b^2 = d$ is equivalent to finding integers a, b satisfying $N(a + b\sqrt{8}) = d$.

From the information given in the problem statement, we have $N(19 + 3\sqrt{8}) = 17^2$ and $N(27 + 5\sqrt{8}) = 23^2$. Therefore, applying Question 3 we have

$$\begin{aligned} 391^2 = 17^2 \cdot 23^2 &= N(19 + 3\sqrt{8})N(27 + 5\sqrt{8}) \\ &= N\big((19 + 3\sqrt{8})(27 + 5\sqrt{8})\big) \\ &= N(633 + 176\sqrt{8}). \end{aligned}$$

Therefore $633^2 - 8(176^2) = 391^2$. It remains to check that 633 and 176 are coprime.

The positive divisors of 633 are $1, 3, 211, 633$. Since $3, 211$, and 633 are not divisors of 176, we can conclude that 1 is the only positive common divisor of 633 and 176.

5. From Question 2, we know $9^2 - 8 \cdot 2^2 = 7^2$. Notice that $N(9+2\sqrt{8}) = 9^2 - 8 \cdot 2^2 = 7$. So, with the result from Question 3 at our disposal, we can repeatedly square $9 + 2\sqrt{8}$ to get elements a_n and b_n of our sequence.

   To that end, define $a_1 = 9$ and $b_1 = 2$. We then recursively define a_n and b_n by

$$a_n + b_n\sqrt{8} = (a_{n-1} + b_{n-1}\sqrt{8})^2 = (a_{n-1}^2 + 8b_{n-1}^2) + 2a_{n-1}b_{n-1}\sqrt{8}.$$

   Therefore $a_n = a_{n-1}^2 + 8b_{n-1}^2$ and $b_n = 2a_{n-1}b_{n-1}$. To see $a_n^2 - 8b_n^2 = 7^{2^n}$ we have

$$a_n^2 - 8b_n^2 = N(a_n + b_n\sqrt{8}) = N((a_1 + b_1\sqrt{8})^{2^{n-1}}) = (N(a_1 + b_1\sqrt{8}))^{2^{n-1}} = 7^{2^n}$$

   where the third equality is obtained by applying Question 3 repeatedly.

   It remains to show that for every n, a_n and b_n are coprime. To this end, first note that a_1 and b_1 are coprime (since $a_1 = 9$ and $b_1 = 2$).

   Next we will prove that if a_n and b_n share a prime factor p (which is equivalent to the statement that a_n and b_n are *not* coprime), then a_{n-1} and b_{n-1} must also share a prime factor p. Once we have proved this, we can repeatedly apply it to show that if there is some n for which a_n and b_n share a prime factor, then a_1 and b_1 must share the same prime factor, a contradiction!

   So, assume p is a prime that divides both a_n and b_n. Since $a_n^2 - 8b_n^2 = 7^{2^n}$, which is odd, a_n and b_n cannot both be even. Therefore $p \neq 2$.

   To proceed, we will repeatedly exploit the following two properties of prime numbers, which we now state without proof:

   - *If p is a prime and p divides ab, then p divides a or p divides b.*
   - *If p is a prime and p divides n^2, then p divides n.*

   The first property is orten called *Euclid's lemma*. Note that the second property is a special case of the first.

   With these facts in our back pocket, let's return to the proof. Since p is an odd prime and p divides $2a_{n-1}b_{n-1}$, we must have that p divides a_{n-1} or b_{n-1}.

   Since p divides a_n, write $pm = a_n$ for some integer m. Suppose first that p divides a_{n-1}, so $pk = a_{n-1}$ for some integer k. Then $pm = p^2k^2 + 8b_{n-1}^2$, which rearranges to $8b_{n-1} = p(m - pk^2)$. Therefore p divides $8b_{n-1}^2$. Since p is an odd prime, we must have that p divides b_{n-1}.

   On the other hand, suppose that p divides b_{n-1} and write $pk = b_{n-1}$. Then similar to the previous case we have $pm = a_{n-1}^2 + 8p^2k^2$. Therefore p also divides a_{n-1}.

   We have proved what we set out to prove: If a_n and b_n share a prime divisor p, then a_{n-1} and b_{n-1} share the prime divisor p. Repeatedly applying this result, we get that if a_n and b_n share a prime divisor p, then $a_1 = 9$ and $b_1 = 2$ share a prime divisor p. However, since 2 and 9 share no prime divisors, a_n and b_n share no prime divisors for all n, completing the proof.

   If you are familiar with induction, you can formalise the above argument as an inductive argument.

To finish things off, let's explicitly calculate the first few terms in our sequences. We have

$$\begin{aligned}
(a_1, b_1) &= (9, 2) \\
(a_2, b_2) &= (113, 36) \\
(a_3, b_3) &= (23\,137, 8\,136) \\
(a_4, b_4) &= (1\,064\,876\,737, 376\,485\,264).
\end{aligned}$$

Sure enough, it turns out that

$$(1\,064\,876\,737)^2 - 8(376\,485\,264)^2 = 33\,232\,930\,569\,601 = 7^{16}.$$

Cool.