Problem of the Month

Problem 8: Some Surprising Squares

May 2025

This month's problem is inspired by Question 9 on the 2025 Euclid contest. The question in its original form is on the next page. As a warm up, try the problem yourself before attempting the Problem of the Month.

While solving Q9(c), you need to find pairs of integers (m, e) satisfying that $m^2 - 8e^2$ is a perfect square. Let's investigate this problem further.

1. For each of the following equations, find one pair (m, e) of non-zero integers that solve it: $m^2 - 8e^2 = 1$, $m^2 - 8e^2 = 4$ and $m^2 - 8e^2 = 9$.

Integers a and b are called *coprime* if they share no positive common divisors other than 1. For example, 3 and 5 are coprime but 4 and 6 are not.

2. Find a pair (m, e) of coprime integers satisfying $m^2 - 8e^2 = 49$.

Let's focus on expressions of the form $a + b\sqrt{8}$, where a and b are integers. Define the *norm* of $a + b\sqrt{8}$ to be $N(a + b\sqrt{8}) = (a + b\sqrt{8})(a - b\sqrt{8})$.

3. Let a, b, c, d be integers. Prove that $N\big((a + b\sqrt{8})(c + d\sqrt{8})\big) = N(a + b\sqrt{8})N(c + d\sqrt{8})$.

4. It turns out that $19^2 - 8(3^2) = 17^2$ and $27^2 - 8(5^2) = 23^2$. Find coprime integers a, b so that $a^2 - 8b^2 = 391^2$.

5. Find infinite sequences of integers $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ satisfying that for all positive integers n,

   - a_n and b_n are coprime, and
   - $a_n^2 - 8b_n^2 = 7^{2^n}$.

   When we write 7^{2^n} we mean $7^{(2^n)}$. So, for example, when $n = 5$, 7^{2^n} is equal to 7^{32} and **not** 49^5.

Here is Question 9 from the 2025 Euclid contest.

Suppose that $p(x) = qx^3 - rx^2 - sx + t$ for some positive integers $q < r < s < t$ which form an arithmetic sequence.

(a) Show that $x = 1$ is a root of $p(x)$.

(b) Suppose that the average of q, r, s, t is 19 and that $p(x)$ has three rational roots. Determine the roots of $p(x)$.

(c) Prove that, for every positive integer $n > 3$, there are at least two arithmetic sequences of positive integers $q < r < s < t$ with common difference $2n$ for which $p(x)$ has three rational roots.
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)