Problem of the Month
Problem 7: Smooth lists
April 2025

Define a function f whose input and output are both lists of n nonnegative integers by

$$f(a_1, a_2, \ldots, a_n) = (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{n-1} - a_n|, |a_n - a_1|)$$

where, as usual, $|x|$ represents the absolute value of x.

For example,
$$f(1, 2, 3, 4) = (|1 - 2|, |2 - 3|, |3 - 4|, |4 - 1|) = (1, 1, 1, 3)$$

and

$$f(2, 3, 5) = (|2 - 3|, |3 - 5|, |5 - 2|) = (1, 2, 3).$$

We will denote by f^k the function that *iterates* the application of f a total of k times. For example,

$$f^4(1, 1, 1, 3) = f^3(0, 0, 2, 2) = f^2(0, 2, 0, 2) = f(2, 2, 2, 2) = (0, 0, 0, 0).$$

We will call a list $(a_1, a_2, \ldots, a_n)$ *smooth* if there is some m for which $f^m(a_1, \ldots, a_n) = (0, 0, \ldots, 0)$. That is, a list is smooth if some number of applications of f will result in the list of all zeros. For example, $(1, 1, 1, 3)$ is smooth since $f^4(1, 1, 1, 3) = (0, 0, 0, 0)$, as demonstrated above.

1. Find a list of length 5 that is not smooth. Find a list of length 7 that is not smooth.

2. Show that for all odd integers $n \geq 1$ there exists a list L of length n that is not smooth.

3. How many smooth lists (a, b, c) are there with a, b, and c each no larger than 100?

4. Suppose L is a list of length 4 consisting of only zeros and ones. Show that L is smooth.

5. Show that all lists of length 4 are smooth.