Problem of the Month

Solution to Problem 6: Regular Polygons and Lattice Points

March 2025

1. Let A and B have coordinates (a, b) and (c, d) respectively. Let α and β be the angles from the positive x-axis to the lines OA and OB respectively, measured in the clockwise direction. For example, if A has coordinates $(1, 1)$ and B has coordinates $(-1, -1)$, $\alpha = 45°$ and $\beta = 215°$.

   Assume towards a contradiction, that a, b, c, and d are integers and $\angle AOB$ is $60°$.

   Case 1, $a \neq 0$ and $c \neq 0$: Since $0 \leq \alpha, \beta < 360°$, then $\beta - \alpha = \pm 60°$ or $\pm 300°$. Therefore, $\tan(\beta - \alpha) = \pm \tan(60°) = \pm\sqrt{3}$. We have $\tan(\alpha) = \frac{b}{a}$ and $\tan(\beta) = \frac{d}{c}$. By the angle sum formula for tan,

   $$\begin{aligned} \tan(\beta - \alpha) &= \frac{\tan(\beta) - \tan(\alpha)}{1 + \tan(\alpha)\tan(\beta)} \\ &= \frac{\frac{d}{c} - \frac{b}{a}}{1 + \left(\frac{b}{a}\right)\left(\frac{d}{c}\right)}. \end{aligned}$$

   Since a, b, c, and d are integers with $a \neq 0$ and $c \neq 0$, $\tan(\beta - \alpha)$ is rational. However, $\sqrt{3}$ is well known to be irrational, which gives us a contradiction. We can now conclude that if A and B are lattice points, then $\angle AOB$ is not $60°$.

   Case 2, $a = 0$ or $c = 0$: In this case, at least one of A or B is on the y-axis. Note that if A is on the y-axis, then B cannot be on the x- or y- axis (since $\angle AOB$ is not $0°$, $90°$, or $180°$). Similarly, if B is on the y-axis, then A cannot be on the x- or y-axis. So actually, exactly one of A or B is on the y-axis, and the other is not on the x-axis.

   Let A' and B' be the result of rotating A and B $90°$ clockwise about the origin. Now exactly one of A' and B' are on the x-axis, and the other is not on the y-axis. Furthermore, $\angle AOB = \angle A'OB'$. We are now back in Case 1 above but with the points A' and B', and can conclude $\angle AOB$ cannot be $60°$.

2. (a) The interior angle of a regular n-gon is $180° - \frac{360°}{n}$, and so $\angle BCD = \angle BAE = 108°$. Since $\triangle EAB$ is isosceles, $\angle ABE = 36°$. Therefore, $\angle CBF = 72°$. Since

   $$180° = 72° + 108° = \angle CBF + \angle BCD,$$

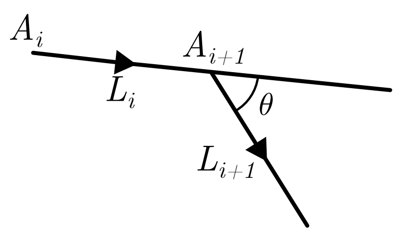
   we have that CD is parallel to BF. We can similarly argue that BC and DF are parallel, and conclude that quadrilateral $FBCD$ is a parallelogram.

   (b) Suppose B, C, and D have coordinates (b_1, b_2), (c_1, c_2), and (d_1, d_2) respectively. Since quadrilateral $FBCD$ is a parallelogram, the difference between the x-coordinates of D and C is equal to the difference between the x-coordinates of F and B. The same holds for the y-coordinates. Therefore, F has coordinates $(b_1 + d_1 - c_1, b_2 + d_2 - c_2)$. We are assuming that B, C, and D are lattice points, so b_1, b_2, c_1, c_2, d_1, and d_2 are all integers. Therefore, the coordinates of F are integers and F is a lattice point.

If you are familiar with vectors, we can rephrase the above argument in terms of vector addition. Since $FBCD$ is a parallelogram, we know $\overrightarrow{CD} + \overrightarrow{CB} = \overrightarrow{CF}$. Therefore, $\vec{D} - \vec{C} + \vec{B} - \vec{C} = \vec{F} - \vec{C}$. This implies $\vec{F} = \vec{D} + \vec{B} - \vec{C}$. Since B, C, and D are lattice points, the corresponding vectors $\vec{B}$, $\vec{C}$, and $\vec{D}$ have integer coordinates. Since $\vec{F}$ is the sum of three vectors with integer coordinates, $\vec{F}$ also has integer coordinates and we can conclude that F is a lattice point.

3. (a) We begin by showing that the angle between L_i and L_{i+1} is $\frac{360°}{n}$ for all i such that $1 \leq i \leq n-1$. The same argument shows that the angle between L_n and L_1 is also $\frac{360°}{n}$.

Let θ be the angle between L_i and L_{i+1}. Then, on the original regular n-gon, we can compute θ by extending the end of L_i past the beginning of L_{i+1} as in the diagram below.



Since the interior angle of a regular n-gon is $180° - \frac{360°}{n}$, we must have

$$\theta = 180° - (180° - \frac{360°}{n}) = \frac{360°}{n}.$$

Let A be the center of $B_1 B_2 \cdots B_n$. Then

$$\frac{360°}{n} = \angle B_1 A B_2 = \angle B_2 A B_3 = \cdots = \angle B_{n-1} A B_n = \angle B_n A B_1.$$

Since each line segment AB_i is the side length of the original regular n-gon, the n triangles $\triangle B_1 A B_2, \triangle B_2 A B_3, \ldots, \triangle B_{n-1} A B_n$, and $\triangle B_n A B_1$ are all congruent.

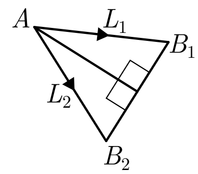
Therefore, the edges of the polygon $B_1 B_2, B_2 B_3, \ldots, B_{n-1} B_n$, and $B_n B_1$ all have equal length.

Consider $\triangle B_1 A B_2$. Since AB_1 and AB_2 have the same length, it is an isosceles triangle. Since $\angle B_1 A B_2 = \frac{360°}{n}$, we have $\angle A B_1 B_2 = \angle A B_2 B_1 = 90° - \frac{180°}{n}$. By the congruence $\triangle B_1 A B_2$ and $\triangle B_2 A B_3$, we have $\angle A B_2 B_3 = \angle A B_3 B_2 = 90° - \frac{180°}{n}$. We can finally compute

$$\angle B_1 B_2 B_3 = \angle B_1 B_2 A + \angle A B_2 B_3 = 90° - \frac{180°}{n} + 90° - \frac{180°}{n} = 180° - \frac{360°}{n}.$$

We can compute every interior angle of the n-gon $B_1 B_2 \cdots B_n$ in the same way to get that all interior angles are equal to $180° - \frac{360°}{n}$. Therefore, $B_1 B_2 \cdots B_n$ is a regular n-gon.

(b) As in the previous solution, let A be the center of the regular n-gon $B_1 B_2 \cdots B_n$, and consider the isosceles triangle $\triangle B_1 A B_2$.



Recall from part (a) that $\angle B_1AB_2 = \frac{360°}{n}$. By drawing a line from the midpoint of B_1B_2 to A, we divide $\triangle B_1AB_2$ into two congruent right-angled triangles. Considering one of these right-angled triangles gives

$$\sin\left(\frac{180°}{n}\right) = \frac{x}{2y}.$$

Therefore,

$$\frac{x}{y} = 2\sin\left(\frac{180°}{n}\right).$$

4. **Case 1, $n = 3$ and $n = 6$:**

   Question 1 tells us that there are no equilateral lattice triangles. In fact, we can also rule out the existence of a regular lattice n-gon where n is a multiple of three. To see this, label the vertices $V_1, \ldots, V_{3n}$ in a clockwise order. Then $\angle V_nV_{2n}V_{3n} = 60°$. Therefore, all three of V_n, V_{2n}, and V_{3n} cannot be lattice points.

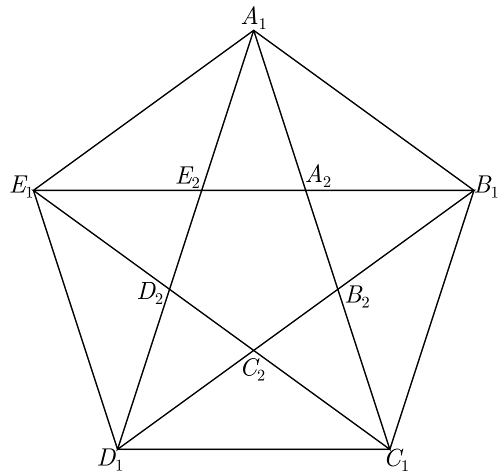
   Case 2, $n = 5$:

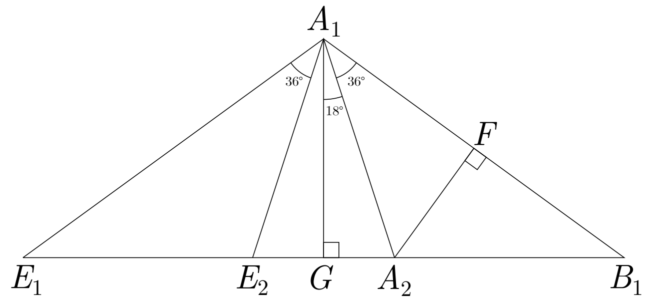
   Next we will show that there are no regular lattice pentagons. We will approach this by showing that if there is a regular lattice pentagon, then we can create a smaller regular lattice pentagon. Then we can do it again, and create an even smaller regular lattice pentagon. We can continue to create smaller and smaller lattice pentagons, until we have a lattice pentagon with side length less than 1, so it cannot be a lattice pentagon! The only way to resolve this contradiction is to conclude that the original pentagon does not exist! Let's execute this plan.

   Consider a regular pentagon $A_1B_1C_1D_1E_1$. Create a smaller regular pentagon by drawing all five diagonals of the pentagon, and taking the five intersections to be the vertices of our smaller pentagon (see the diagram below). More precisely,

   - the lines E_1B_1 and A_1C_1 intersect at A_2,
   - the lines A_1C_1 and B_1D_1 intersect at B_2,
   - the lines B_1D_1 and C_1E_1 intersect at C_2,
   - the lines C_1E_1 and D_1A_1 intersect at D_2, and
   - the lines D_1A_1 and E_1B_1 intersect at E_2.

   Then $A_2B_2C_2D_2E_2$ is a regular pentagon (see if you can prove this!). Suppose the side length of pentagon $A_1B_1C_1D_1E_1$ is l_1, and the side length of $A_2B_2C_2D_2E_2$ is l_2. The goal now is to write down an expression for l_2 in terms of l_1.





By the solution to Question 2, $\angle E_1A_1E_2 = \angle B_1A_1A_2 = 36°$. Since the interior angle of a regular pentagon is $108°$, we have $\angle E_2A_1A_2 = 108° - 36° - 36° = 36°$.

As in the image above, let G be a point on E_2A_2 so that GA_1 is perpendicular to E_2A_2. Let F be the point on A_1B_1 so that FA_2 is perpendicular to A_1B_1. Then G and F are the midpoints of E_2A_2 and A_1B_1 respectively (this fact needs proof, but I will leave it up to you!).

There are now two right-angled triangles, $\triangle A_1FA_2$ and $\triangle A_1GA_2$. Consider the former triangle. Note that the length of A_1F is $\frac{1}{2}l_1$ and $\angle A_2A_1F = 36°$. Let x be the length of A_1A_2. Then

$$\frac{l_1}{2x} = \cos(36°).$$

Now consider right-angled triangle $\triangle A_1GA_2$. We have $\angle GA_1A_2 = 18°$, and the length of GA_2 is $\frac{1}{2}l_2$. Therefore,

$$\sin(18°) = \frac{l_2}{2x}.$$

Solving for x in both of these equations and rearranging gives

$$l_2 = l_1\frac{\sin(18°)}{\cos(36°)}.$$

At the end of this solution, there is an extra section which shows how to deduce that $\sin(18°) = \frac{\sqrt{5}-1}{4}$ and $\cos(36°) = \frac{\sqrt{5}+1}{4}$. Using these exact values we have

$$l_2 = \frac{\sqrt{5}-1}{\sqrt{5}+1}l_1 = \frac{3-\sqrt{5}}{2}l_1.$$

Note that $2 < \sqrt{5} < 3$ and so $0 < \frac{3-\sqrt{5}}{2} < 1$.

Great, we can now repeat this process of creating smaller and smaller pentagons, and we can compute the side length of each one as follows.

Let k be a positive integer, and suppose $A_kB_kC_kD_kE_k$ is a regular pentagon with side length l_k. Create the pentagon $A_{k+1}B_{k+1}C_{k+1}D_{k+1}E_{k+1}$ as above by declaring that

- the lines E_kB_k and A_kC_k intersect at A_{k+1},
- the lines A_kC_k and B_kD_k intersect at B_{k+1},

- the lines B_kD_k and C_kE_k intersect at C_{k+1},
- the lines C_kE_k and D_kA_k intersect at D_{k+1}, and
- the lines D_kA_k and E_kB_k intersect at E_{k+1}.

Then $A_{k+1}B_{k+1}C_{k+1}D_{k+1}E_{k+1}$ is a regular pentagon with side length l_{k+1} where

$$l_{k+1} = \frac{3-\sqrt{5}}{2} l_k.$$

Repeatedly applying the equation $l_{k+1} = \frac{3-\sqrt{5}}{2} l_k$ we have that for any positive integer k,

$$l_k = \left(\frac{3-\sqrt{5}}{2}\right)^{k-1} l_1.$$

We will now prove that there is some k large enough so that $l_k < 1$. Choose a positive integer k so that

$$k > \frac{-\log(l_1)}{\log(3-\sqrt{5}) - \log(2)} + 1.$$

Here we don't care what the base is for the logs, so we will be lazy and not write anything down as the base. After rearranging the inequality a little, and applying some log laws we have

$$k - 1 > \frac{\log\left(\frac{1}{l_1}\right)}{\log\left(\frac{3-\sqrt{5}}{2}\right)}.$$

Since $\frac{3-\sqrt{5}}{2} < 1$, $\log\left(\frac{3-\sqrt{5}}{2}\right) < 0$. Therefore,

$$\begin{aligned}
(k-1)\log\left(\frac{3-\sqrt{5}}{2}\right) &< \log\left(\frac{1}{l_1}\right) \\
\Rightarrow \quad \left(\frac{3-\sqrt{5}}{2}\right)^{k-1} &< \frac{1}{l_1} \\
\Rightarrow \quad l_1\left(\frac{3-\sqrt{5}}{2}\right)^{k-1} &< 1 \\
\Rightarrow \quad l_k &< 1.
\end{aligned}$$

Great! Let's put everything together. Suppose $A_1B_1C_1D_1E_1$ is a regular lattice pentagon with length l_1. Then by Question 2(b), the regular pentagon $A_kB_kC_kD_kE_k$ is a lattice pentagon for all positive integers k. However, when

$$k > \frac{-\log(l_1)}{\log(3-\sqrt{5}) - \log(2)} + 1$$

we have shown that the side length l_k of $A_kB_kC_kD_kE_k$ satisfies $l_k < 1$. Since the smallest distance between two lattice points in the plane is 1, $A_kB_kC_kD_kE_k$ cannot be a lattice pentagon, which is a contradiction! Therefore, a regular pentagon cannot be a lattice pentagon.

Case 3, $n \geq 7$:
The general strategy for this case will be the same as in the case $n = 5$. The difference here will be our construction of successive smaller regular polygons.

Let P_1 be a regular n-gon with side length l_1. Let P_k be the regular n-gon obtained from P_1 by applying k times the process from Question 3. Let P_k have side length l_k. Then from the solution to 3(b) above we have

$$l_k = l_1 \left(2 \sin\left(\frac{180^\circ}{n}\right)\right)^k.$$

Between 0° and 90°, the sine function is strictly increasing. Therefore, for $n \geq 7$ we have

$$0 < 2\sin\left(\frac{180^\circ}{n}\right) < 2\sin\left(\frac{180^\circ}{6}\right) = 1.$$

As in the $n = 5$ case, we want to show that if we choose k to be big enough, $l_k < 1$. To that end, let $a = 2\sin\left(\frac{180^\circ}{n}\right)$ and choose a positive integer k so that

$$k > \frac{-\log(l_1)}{\log(a)}.$$

Then again, since $0 < a < 1$, $\log(a) < 0$ and we have

$$\begin{aligned} k\log(a) &< \log\left(\frac{1}{l_1}\right) \\ \Rightarrow \qquad a^k &< \frac{1}{l_1} \\ \Rightarrow \qquad l_1 a^k &< 1 \\ \Rightarrow \qquad l_k &< 1. \end{aligned}$$

It remains to show that if P_k is a lattice polygon, then so is P_{k+1}. Let P_k be the lattice polygon $A_1A_2\cdots A_n$, where A_i has coordinates (a_i, b_i). Now, translate the polygon P_{k+1} so that its center is at the origin O, with coordinates $(0, 0)$. Let P_{k+1} be the polygon $B_1B_2\cdots B_n$. Then for each $i < n$, B_i has coordinates $(a_{i+1} - a_i, b_{i+1} - b_i)$, and B_n has coordinates $(a_1 - a_n, b_1 - b_n)$. Since each of the a_i and b_i are integers, we have that each of the B_i is a lattice point and P_{k+1} is a lattice polygon.

Great, now we can put everything together. Suppose P_1 is a regular lattice n-gon with side length l_1. Then for each positive integer k, we can create another regular lattice n-gon with side length $l_k = l_1 a^k$, where $a = 2\sin\left(\frac{180^\circ}{n}\right)$. If $k > \frac{-\log(l_1)}{\log(a)}$, then $l_k < 1$, contradicting the fact that P_k is a lattice polygon.

Therefore, we can conclude that for $n \geq 7$, there is no regular lattice n-gon.

Through the cases we have ruled out the existence of a regular lattice n-gon for all $n \geq 3$ except for $n = 4$. Of course, there are plenty of lattice squares!

There are a couple of things worth discussing about the solutions above.

1. With a little bit of love and care, the solution to Question 1 can be massaged to show that if A, B, and C are distinct lattice points, and if $\theta = \angle ABC$, then $\tan(\theta)$ is a rational

number. There is a theorem called *Niven's Theorem* which says the following:

Niven's Theorem: *Let* $\theta = (180^\circ)\left(\frac{a}{b}\right)$*, where* a *and* b *are integers. If* $\tan(\theta)$ *is rational, then* $\frac{a}{b}$ *is an integer or* $\frac{a}{b} = \frac{2k+1}{4}$ *where* k *is an integer.*

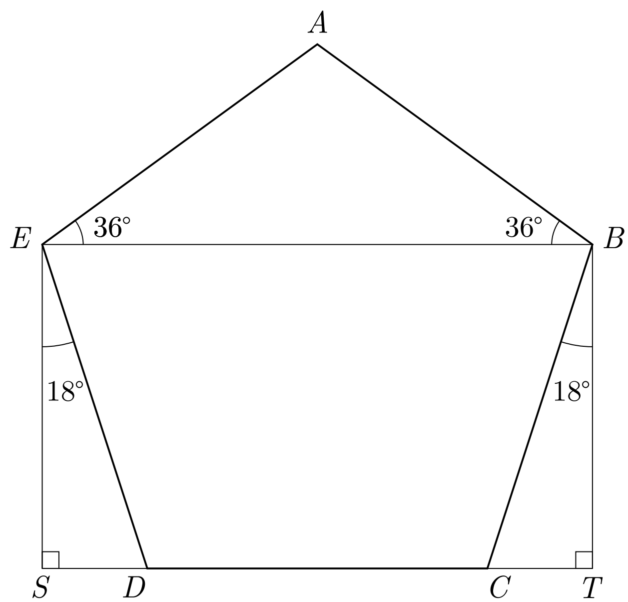
This theorem can be used to rule out the existence of regular lattice n-gons for all n except for $n = 8$. The case of regular lattice octagons can then be dealt with separately.

2. When dealing with the case $n = 5$ and the case $n \geq 7$ in the solution to Question 4, we obtained an infinite sequence of positive numbers $l_1, l_2, l_3, \ldots$ with the property that $l_i > l_{i+1}$ for all positive integers i. We needed to show that there is some k large enough so that $l_k < 1$. The fact that the sequence $l_1, l_2, l_3, \ldots$ is decreasing *does not* guarantee that the sequence eventually becomes smaller than 1. To see this, consider the sequence $1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, \ldots$. This is a sequence of positive numbers that is decreasing, but is never less than 1. This is why we had to go through so much trouble to find an explicit k and prove that $l_k < 1$.

A computation of $\sin(18^\circ)$ and $\cos(36^\circ)$

Here we will prove that $\sin(18^\circ) = \frac{1}{4}(\sqrt{5} - 1)$ and $\cos(36^\circ) = \frac{1}{4}(\sqrt{5} + 1)$.

Consider the regular pentagon $ABCDE$ with side length 1 as shown in the diagram below.



Extend the line DC in both directions and let S and T be points on the extended line so that ES and BT are perpendicular to DC. From our solution to Question 2(a), we know ST is parallel to EB, and therefore, the length of EB is equal to the length of ST.

In our solution to Question 2(a) we showed that $\angle AEB = \angle ABE = 36^\circ$. Therefore the length of EB is $2\cos(36^\circ)$.

Since $\angle EDC = \angle BCD = 108^\circ$, we have $\angle EDS = \angle BCT = 72^\circ$ and so $\angle SED = \angle CBT = 18^\circ$. Then the length of ST is the sum of the lengths of SD, DC, and CT. The length of DC is 1, and the lengths of SD and CT are both $\sin(18^\circ)$. Since the length of EB is equal to the length of ST we have

$$2\cos(36^\circ) = 2\sin(18^\circ) + 1.$$

By the double angle formula for cosine, we have

$$2(1 - 2(\sin(18^\circ))^2) = 2\sin(18^\circ) + 1.$$

If we let $x = \sin(18^\circ)$ we have that x satisfies

$$4x^2 + 2x - 1 = 0.$$

The quadratic formula then gives us

$$x = \frac{-1 \pm \sqrt{5}}{4}.$$

Since $\sin(18^\circ) > 0$ we must have $\sin(18^\circ) = \frac{1}{4}(\sqrt{5} - 1)$. Using the equation

$$2\cos(36^\circ) = 2\sin(18^\circ) + 1$$

gives us

$$\cos(36^\circ) = \frac{\sqrt{5} - 1}{4} + \frac{1}{2} = \frac{\sqrt{5} + 1}{4}.$$