Problem of the Month

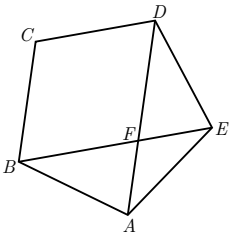
Problem 6: Regular Polygons and Lattice Points

March 2025

A *lattice point* in the Cartesian plane is a point (a, b) with the property that both a and b are integers. In this Problem of the Month, we will investigate regular polygons that have every vertex lying on a lattice point.

1. Let A and B be distinct lattice points on the Cartesian plane, neither of which have coordinates $(0, 0)$. Show that the measure of $\angle AOB$ cannot equal $60°$, where O has coordinates $(0, 0)$.

2. Consider a regular pentagon $ABCDE$. Let F be the point of intersection of lines AD and BE.

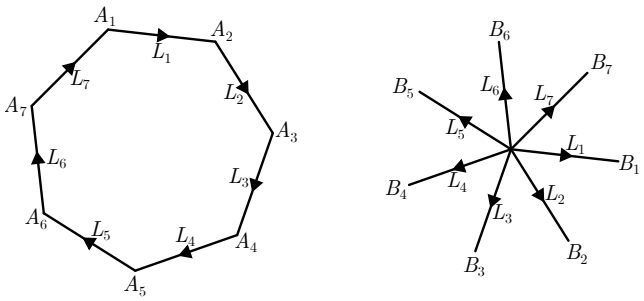


   (a) Show that the quadrilateral $FBCD$ is a parallelogram.

   (b) Show that if B, C, and D are lattice points then so is F.

3. View a regular n-gon as a collection of n line segments. Give each line segment a direction (indicated by an arrow), moving in a clockwise direction (see the image below). Label the line segments $L_1, L_2, \ldots, L_n$. For each i, label the starting point of L_i by A_i. Note that the points $A_1, A_2, \ldots, A_n$ are the vertices of the n-gon.

   Now, translate the line segments (without any rotation) so that the points A_i all coincide. For each L_i, label its new endpoint by B_i. Below is an image of this process when $n = 7$.



   (a) Show that the polygon $B_1B_2 \cdots B_n$ is a regular n-gon.

   (b) Let y be the length of L_1 and x be the length of the line segment B_1B_2. Compute $\frac{x}{y}$ in terms of n.

4. We call a polygon that has every vertex lying on a lattice point a *lattice polygon*. Show that if a regular n-gon is a lattice polygon, then $n = 4$.