Problem of the Month

Problem 6: Regular Polygons and Lattice Points

March 2025

Hint

1. Choose some lattice points A and B and compute $\tan(\angle AOB)$. The values of $\tan(\angle AOB)$ are limited to some subset of the real numbers. Can you figure out what this subset is, and can you prove that $\tan(60°)$ is not in that subset?

2. (a) Compute the interior angle of a regular pentagon. Then compute $\angle CBF$. Is there a theorem from geometry you can use to prove that two lines are parallel?

   (b) Suppose X and Y are points with coordinates (x_1, x_2) and (y_1, y_2). Let Z be the point with coordinates $(x_1 + y_1, x_2 + y_2)$. Plot the four points O, X, Y, Z on the Cartesian plane. Is there anything special about the quadrilateral with vertices O, X, Y, Z? Try it with some specific points X and Y.

3. (a) Start by computing the angle between L_1 and L_2.

   (b) Let A be the center of the n-gon $B_1B_2 \cdots B_n$ (that is, A is the point where all the lines L_i meet in the second image in the statement of the problem). Then $\triangle AB_1B_2$ is an isosceles triangle with one of its side lengths equal to x, and another one equal to y.

4. Use Question 1 to rule out the existence of regular lattice triangles and hexagons. Use Question 2 to rule out the existence of regular lattice pentagons. Use Question 3 to rule out the existence of regular lattice n-gons where $n \geq 7$.

   As a general strategy, assume that there is a regular lattice n-gon, and try to construct a smaller regular lattice n-gon. If you can do this once, then you can do it again and again. Is it a problem to have smaller and smaller lattice polygons? Is this even possible?