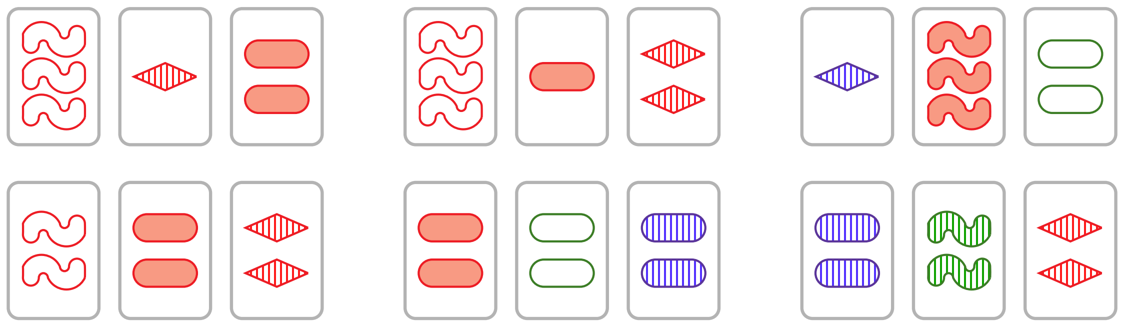
Problem of the Month

Solution to Problem 5: SET!

February 2025

1. The six sets are:



   Going by the labels in the statement of the problem, the six sets are

$$\{A, B, E\}, \{A, F, L\}, \{C, G, I\}, \{D, E, L\}, \{E, I, J\}, \{J, K, L\}.$$

2. Since each property has three options, and there are four properties, there are

$$3 \times 3 \times 3 \times 3 = 81$$

   cards in a SET! deck.

For the next two questions, we first make the following observation: Given any two cards A and B, there is a unique third card C so that the three cards A, B, and C form a set. Let's see why this is true.

For each property, if the options on A and B are the same, then the option for that property on card C must also be the same as it is for A and B. If the options are not the same for A and B, then C must have the third option for that property.

For example, if A and B are both shaded solid, then C must be shaded solid. On the other hand, if the shading of A is striped and the shading of B is solid, then the shading of C must be open.

3. With the observation above in mind, to create a set that contains the given card, we just have to choose any other card. Then the given card with our choice of second card uniquely determines a third card that forms a set.

   There are 80 choices for a second card. However this counts every set twice! To see why, call the card we are given in the question A. Suppose we choose a card B, and the unique third card that forms a set is C. If instead we choose C as the second card, then B is the unique card that forms a set with A and C. So choosing B as the second card results in the same set as choosing C as the second card.

   Therefore the answer is $\frac{80}{2} = 40$.

4. **Solution 1:** We again rely on the observation above that once we have chosen two cards as part of a set, the third is determined.

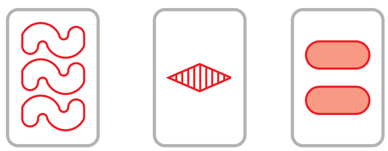
   There are 81 ways to choose the first card in our set, and 80 ways to choose the second. However, since there are $3! = 6$ ways to order the three cards in a set, the number 81×80 counts each set six times. Therefore the number of different sets in a SET! deck is $\frac{81 \times 80}{6} = 1080$.

   Solution 2: Using Question 3, we know each card appears in 40 sets. Since there are 81 cards in a full deck, and 3 cards in a set, there are $\frac{40 \times 81}{3} = 1080$ sets in a SET! deck.

5. For this question, we will need to introduce a different way of thinking about the cards. We can encode each card by a 4-tuple (q, r, s, t), where each entry q, r, s, t is either 1, 2, or 3 as in the following table:

Entry in the 4-tuple	Number	Shading	Colour	Shape
1	1	open	red	diamond
2	2	striped	green	oval
3	3	solid	purple	squiggle

   For example, consider the cards from one of the sets from Problem 1 above:



   These cards are encoded by the tuples $(3, 1, 1, 3)$, $(1, 2, 1, 1)$, and $(2, 3, 1, 2)$ respectively.

   Let's think about what it takes for three 4-tuples to form a set. Each coordinate in the tuple (ie, the first, second, third, or fourth entry) corresponds to one of the properties (number, shading, colour, and shape, in that order).

   The condition that a property has the same option across the three cards translates to the entry in the corresponding coordinate being the same across all three 4-tuples. In the example above, all three cards are red. Therefore, the entries in the third coordinates of all three 4-tuples are 1.

   The condition that a property has all different options across the three cards translates to each of 1, 2, and 3 appearing in the corresponding coordinate in the three 4-tuples. In the example above, all three cards have different shadings. Therefore, the entries 1, 2, and 3 appear in some order in the second coordinates of the three 4-tuples.

   Our goal now is to characterise exactly when a collection of three numbers $\{a, b, c\}$ is either $\{1, 1, 1\}$, $\{2, 2, 2\}$, $\{3, 3, 3\}$, or $\{1, 2, 3\}$. Note that in the last case, a, b, c must be equal to $1, 2, 3$ in some order (not necessarily $a = 1$, $b = 2$, and $c = 3$).

   Claim: Suppose a, b, c are three integers, each of which is either 1, 2, or 3. Then $a+b+c$ is a multiple of three exactly when either $a = b = c$ or a, b, and c are all distinct.

   Let's justify this claim. First, if $a = b = c$ then $a + b + c$ is equal to 3, 6, or 9. If a, b, and c are distinct, then a, b, and c are 1, 2, and 3 in some order. Therefore,

$a+b+c = 1+2+3 = 6$. Now suppose it is not true that all three of a, b, and c are equal or distinct. That means two of the numbers are equal, and one is not. We can check the sum $a+b+c$ in each of these cases.

$$1+1+2 = 4$$
$$1+1+3 = 5$$
$$2+2+1 = 5$$
$$2+2+3 = 7$$
$$3+3+1 = 7$$
$$3+3+2 = 8.$$

In all of these cases, the sum $a+b+c$ is not a multiple of 3, so the claim is true!

Great! Let's harness this by adding up coordinates among tuples to check whether or not three 4-tuples come from cards that form a set.

To make our lives a little easier, given two 4-tuples, let's add them together to create another 4-tuple by simply adding up the coordinates. More precisely, given two 4-tuples $\mathbf{v} = (a, b, c, d)$ and $\mathbf{w} = (p, q, r, s)$, we define

$$\mathbf{v} + \mathbf{w} = (a+p, b+q, c+r, d+s).$$

If you have experience with vectors, you may recognise that this is exactly how we add two vectors together. Using this definition of addition for 4-tuples, we can repeatedly add as many 4-tuples together as we like!

In particular, given three 4-tuples we can add them together to get another 4-tuple. The resulting 4-tuple may include entries other than 1, 2, and 3, but that's okay! By using the claim above, we know that the three 4-tuples correspond to three cards that form a set exactly when the 4-tuple resulting from adding them together has the property that every entry is a multiple of 3. Check for yourself that this works for the example set at the beginning of the solution to Problem 5.

More precisely, suppose $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$ are three 4-tuples corresponding to SET! cards. Then $\mathbf{v} + \mathbf{w} + \mathbf{u}$ has every coordinate a multiple of 3 exactly when the cards corresponding to $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$ form a set. Choose your favourite set, and your favourite non-set, and check this for yourself!

We are now ready to attack the original question. First convert the 81 cards in a SET! deck into 4-tuples. For each coordinate, the integers 1, 2, and 3 appear exactly 27 times each (this is because once the entry in a coordinate has been fixed, there are three different options for each of the remaining three coordinates, and $3 \times 3 \times 3 = 27$). Therefore, for each coordinate, the sum over all 81 entries in that coordinate is $27 \times 1 + 27 \times 2 + 27 \times 3 = 162$, which is a multiple of 3.

Considering the question at hand, we have collected 26 sets from a full SET! deck. We want to show that the remaining three cards also form a set. Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be the three 4-tuples corresponding to the remaining three cards. For the 78 cards used in the 26 sets, label the 78 4-tuples $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{78}$ so that for every positive integer $k \leq 26$, $\mathbf{v}_{3k-2}, \mathbf{v}_{3k-1}$, and $\mathbf{v}_{3k}$ form a set (ie, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a set, $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$ form a set and so on).

Then we know that for every positive integer $k \leq 26$, $\mathbf{v}_{3k-2} + \mathbf{v}_{3k-1} + \mathbf{v}_{3k}$ is a 4-tuple where every coordinate is a multiple of 3. Summing up all of the resulting 4-tuples gives us that

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_{78} = (a, b, c, d)$$

where a, b, c, d are all multiples of 3. We also know

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_{78} + \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 = (162, 162, 162, 162).$$

We can now conclude that

$$\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 = (162 - a, 162 - b, 162 - c, 162 - d).$$

All of 162, a, b, c, and d are all multiples of 3. Therefore, $162 - a$, $162 - b$, $162 - c$, and $162 - d$ are multiples of 3. We can finally conclude that $\mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_3$ are 4-tuples corresponding to three cards that form a set.