Problem of the Month
Problem 4: A Polynomial Sandwich
January 2025

Let a, b, c, and d be rational numbers and $f(x) = ax^3 + bx^2 + cx + d$. Suppose $f(n)$ is an integer whenever n is an integer and that

$$\frac{1}{3}n^3 - n - \frac{2}{3} \le f(n) \le \frac{1}{3}n^3 + n^2 + 2n + \frac{4}{3}$$

for every integer n with the possible exception of $n = -2$.

1. Show that $a = \dfrac{1}{3}$.

2. Find $f(10^{2025}) - f(10^{2025} - 1)$.