Problem of the Month

Problem 3: Multiplication in two dimensions

December 2025

This month's problem is inspired by Question B2(c) on the 2025 Canadian Senior Mathematics Contest. Here is the original question:

> The points X, Y, and Z have coordinates $X(0,0)$, $Y(7,24)$, and $Z(15,0)$. The point W is on the line segment YZ such that $\angle WXZ = 3\angle WXY$. Determine the coordinates of W.

Give it a try as a warm up before reading on!

One path to solving this problem is to introduce a point V on the line segment YZ so that $\angle VXZ = \angle VXY$. If you do this, you find the coordinates of V are $(12,9)$. These are very nice numbers! In this month's POTM, we will investigate how the points X, Y, and Z were chosen so that the coordinates of V (and eventually W) are nice rational numbers. This will involve a way to multiply points on the Cartesian plane.

1. Given points (a,b) and (c,d) on the Cartesian plane, define their *product* as

   $$(a,b) * (c,d) = (ac - bd, ad + bc).$$

   So, for example, $(3,5) * (2,-1) = (3 \cdot 2 - 5 \cdot (-1), 3 \cdot (-1) + 5 \cdot 2) = (11,7)$.

   (a) Find a point (a,b) satisfying $(1,1) * (a,b) = (0,-2)$.

   (b) Find a point (a,b) with the property that $(a,d) * (c,d) = (c,d)$ for every point (c,d) in the Cartesian plane.

2. For a point (a,b) in the Cartesian plane, denote by $(a,b)^k$ the point obtained by taking the product of (a,b) with itself k times. Compute $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^{2025}$.

3. Denote by O the origin $(0,0)$, and by E the point with coordinates $(1,0)$. Let A be a point in the Cartesian plane. Define $|A|$ to be the distance from A to O. Define $\theta(A)$ to be the measure of the angle $\angle EOA$, measured counterclockwise around O from the line segment OE. For example, $|(-\sqrt{2}, -\sqrt{2})| = \sqrt{(-2)^2 + (-2)^2} = 2$ and $\theta(-\sqrt{2}, -\sqrt{2}) = 225°$.

   (a) Compute $|(0,2) * (1,1)|$ and $\theta((0,2) * (1,1))$.

   (b) Let D_1 and D_2 be points with $|D_1| = r_1$, $|D_2| = r_2$, $\theta(D_1) = \phi_1$ and $\theta(D_2) = \phi_2$. Compute $|D_1 * D_2|$ and $\theta(D_1 * D_2)$ in terms of r_1, r_2, ϕ_1, and ϕ_2.

4. The point $(2,3)$ satisfies the equation $x^4 = (-119, -120)$ since $(2,3)^4 = (-119, -120)$. Find three other points satisfying $x^4 = (-119, -120)$.

5. Let Y have coordinates $(7,24)$. Find a point F in the Cartesian plane with integer coordinates so that $OF^2 = OY$ and $2\angle EOF = \angle EOY$.