Problem of the Month

Problem 1: Displacing Permutations

October 2025

In this problem, X_n will denote the set of integers $\{1, 2, 3, \ldots, n\}$. A *permutation* of X_n is an ordered list of these integers that contains each of them exactly once. For example, there are exactly 6 permutations of X_3, and they are

$$123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321$$

Another way to think about a permutation is as a function from the set X_n to itself, where no two inputs to the function have the same output. For example, the permutation 213 of X_3 represents the function that sends 1 to 2, sends 2 to 1, and sends 3 to itself. The permutation σ of X_5 denoted by 43512 satisfies $\sigma(1) = 4$, $\sigma(2) = 3$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 2$.

In this problem, the *displacement* of a permutation σ of X_n is equal to $\mathbf{D}(\sigma) = \sum_{i=1}^{n} |i - \sigma(i)|$. For example, with σ from the paragraph above, we have

$$\begin{aligned}\mathbf{D}(\sigma) &= |1 - \sigma(1)| + |2 - \sigma(2)| + |3 - \sigma(3)| + |4 - \sigma(4)| + |5 - \sigma(5)| \\ &= |1 - 4| + |2 - 3| + |3 - 5| + |4 - 1| + |5 - 2| \\ &= 3 + 1 + 2 + 3 + 3 = 12\end{aligned}$$

1. Suppose $n \geq 2$. Determine the number of permutations σ of X_n that satisfy $\mathbf{D}(\sigma) = 2$.
2. Suppose $n \geq 4$. Determine the number of permutations σ of X_n that satisfy $\mathbf{D}(\sigma) = 4$.
3. Prove for all $n \geq 2$, if σ is a permutation of X_n, then $\mathbf{D}(\sigma)$ is even.
4. Given an odd positive integer n and a permutation σ of X_n, determine the maximum possible value of $\mathbf{D}(\sigma)$.
5. Given an odd positive integer n, determine the number of permutations σ of X_n have the property that $\mathbf{D}(\sigma)$ is equal to the maximum from the previous question.