Problem of the Month

Problem 0: Equations in the integers

September 2025

Suppose a, b, and c are positive integers. In this problem, a *non-negative solution* to the equation $ax + by = c$ is a pair $(x, y) = (u, v)$ of integers with $u \geq 0$ and $v \geq 0$ satisfying $au + bv = c$. For example, $(x, y) = (7, 0)$ and $(x, y) = (3, 3)$ are non-negative solutions to $3x + 4y = 21$, but $(x, y) = (-1, 6)$ is not.

1. Determine all non-negative solutions to $5x + 8y = 120$.
2. Determine the largest positive integer c with the property that there is no non-negative solution to $5x + 8y = c$.

In Questions 3, 4, and 5, a and b are assumed to be positive integers satisfying $\gcd(a, b) = 1$.

3. Determine the largest non-negative integer c with the property that there is no non-negative solution to $ax + by = c$. The value of c should be expressed in terms of a and b.
4. Determine the number of non-negative integers c for which there are exactly 2025 non-negative solutions to $ax + by = c$. As with Question 3, the answer should be expressed in terms of a and b.
5. Suppose $n \geq 1$ is an integer. Determine the sum of all non-negative integers c for which there are exactly n nonnegative solutions to $ax + by = c$. The answer should be expressed in terms of a, b, and n.

Fact: You may find it useful that for integers a and b with $\gcd(a, b) = 1$, there always exist integers x and y such that $ax + by = 1$, though x and y may not be non-negative.