The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2025 Hypatia Contest

Thursday, April 3, 2025
(in North America and South America)

Friday, April 4, 2025
(outside of North America and South America)

Solutions

1. (a) After 7 minutes, Abi has a total of $30 + (7 \times 2) = 30 + 14 = 44$ tokens.

   (b) After 12 minutes, Desiree had received $12 \times 2 = 24$ additional tokens. After 12 minutes, she had 37 tokens in total, and so Desiree started with $37 - 24 = 13$ tokens.
   A total of 80 tokens were initially distributed, and so Carl started with $80 - 13 = 67$ tokens.

   (c) At the start, Essi received 12 tokens and Francis received $100 - 12 = 88$ tokens.
   After t minutes, Essi has $12 + 2t$ tokens and Francis has $88 + 2t$ tokens.
   After t minutes, Francis has 3 times as many tokens as Essi, and so $88 + 2t = 3(12 + 2t)$.
   Solving for t, we get $88 + 2t = 36 + 6t$ or $52 = 4t$, and so $t = \frac{52}{4} = 13$.

2. (a) *Solution 1*

   Since 10% of 5 cm is $\frac{10}{100} \times 5 \text{ cm} = 0.1 \times 5 \text{ cm} = 0.5 \text{ cm}$, then the length of the resulting rectangle is $5 \text{ cm} + 0.5 \text{ cm} = 5.5 \text{ cm}$, and its area is $5.5 \text{ cm} \times 4 \text{ cm} = 22 \text{ cm}^2$.

   Solution 2

   When 5 cm is increased by 10%, the resulting length is $\left(1 + \frac{10}{100}\right) \times 5 \text{ cm}$ or $1.1 \times 5 \text{ cm}$ which is equal to 5.5 cm.
   Thus, the area of the resulting rectangle is $5.5 \text{ cm} \times 4 \text{ cm} = 22 \text{ cm}^2$.

   (b) *Solution 1*

   A square with area 100 cm^2 has both length and width equal to $\sqrt{100 \text{ cm}^2} = 10 \text{ cm}$.
   Since 30% of 10 cm is $\frac{30}{100} \times 10 \text{ cm} = 0.3 \times 10 \text{ cm} = 3 \text{ cm}$, then the length of the resulting rectangle is $10 \text{ cm} + 3 \text{ cm} = 13 \text{ cm}$, and its width is $10 \text{ cm} - 3 \text{ cm} = 7 \text{ cm}$.
   The area of the resulting rectangle is $13 \text{ cm} \times 7 \text{ cm} = 91 \text{ cm}^2$ which is $\frac{91 \text{ cm}^2}{100 \text{ cm}^2} \times 100\% = 91\%$ of the area of the original square.
   Therefore, the area decreased by $100\% - 91\% = 9\%$.

   Solution 2

   A square with area 100 cm^2 has both length and width equal to $\sqrt{100 \text{ cm}^2} = 10 \text{ cm}$.
   When 10 cm is increased by 30%, the resulting length is $\left(1 + \frac{30}{100}\right) \times 10 \text{ cm}$ or $1.3 \times 10 \text{ cm}$ which is equal to 13 cm.
   When 10 cm is decreased by 30%, the resulting length is $\left(1 - \frac{30}{100}\right) \times 10 \text{ cm}$ or $0.7 \times 10 \text{ cm}$ which is equal to 7 cm.
   Thus, the area of the resulting rectangle is $13 \text{ cm} \times 7 \text{ cm} = 91 \text{ cm}^2$ which is $100 \text{ cm}^2 - 91 \text{ cm}^2 = 9 \text{ cm}^2$ less than the area of the original square.
   Therefore, the area decreased by $\frac{9 \text{ cm}^2}{100 \text{ cm}^2} \times 100\% = 9\%$.

   (c) Suppose the length of the original rectangle is ℓ and its width is w.
   Then the length of the resulting rectangle is $\left(1 + \frac{x}{100}\right) \times \ell$, and its width is $\left(1 - \frac{20}{100}\right) \times w$ or $\frac{8}{10}w$.
   The area of the original rectangle, ℓw, is equal to the area of the resulting rectangle $\left(1 + \frac{x}{100}\right) \times \ell \times \frac{8}{10}w$.

Setting the areas equal and simplifying, we get the following equivalent equations:

$$\begin{aligned}
\left(1+\frac{x}{100}\right)\times \ell \times \frac{8}{10}w &= \ell w \\
\left(1+\frac{x}{100}\right)\times \frac{8}{10} \times \ell w &= \ell w \\
\left(1+\frac{x}{100}\right)\times \frac{8}{10} &= 1 \quad (\text{since } \ell w > 0) \\
1+\frac{x}{100} &= \frac{10}{8} \\
\frac{x}{100} &= \frac{10}{8}-\frac{8}{8} \\
\frac{x}{100} &= \frac{2}{8} \\
x &= \frac{1}{4}\times 100
\end{aligned}$$

and so $x = 25$.

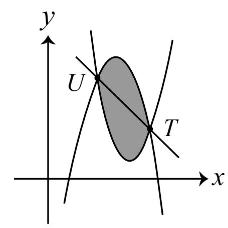
3. (a) We begin by determining where the parabola and the line intersect. Setting the right-hand sides of the equations equal and solving, we get

$$\begin{aligned}
x^2+3x-12 &= 2x \\
x^2+x-12 &= 0 \\
(x+4)(x-3) &= 0
\end{aligned}$$

and so $x = -4$ and $x = 3$.
Applying the given formula with $p = 3$ and $q = -4$, the area enclosed by the parabola and the line is $\dfrac{(3-(-4))^3}{6} = \dfrac{7^3}{6} = \dfrac{343}{6}$.

(b) Suppose the x-coordinates of V and W are the integers v and w respectively, with $v > w$. Then v and w are the solutions of the equation $mx - 6 = -x^2 + 7x - 90$, which when simplified is $x^2 + (m-7)x + 84 = 0$.
Therefore, the equation $x^2 + (m-7)x + 84 = 0$ is equivalent to $(x-v)(x-w) = 0$ or $x^2 - (v+w)x + vw = 0$. Thus, the product of the roots, vw, is equal to 84.
The area enclosed by the parabola and the line is equal to $\dfrac{(v-w)^3}{6}$, which is as small as possible when $v - w$ is as small as possible.
To determine the smallest possible enclosed area, we must determine the smallest possible value of $v - w$ for integers v and w with $vw = 84$ and $v > w$.
The positive factor pairs (w, v) of 84 are $(1, 84)$, $(2, 42)$, $(3, 28)$, $(4, 21)$, $(6, 14)$, and $(7, 12)$.
The smallest possible value of $v - w$ is equal to 5, which occurs when $v = 12$ and $w = 7$.
Each of the factors listed can also be negative, from which we similarly determine that the smallest possible value of $v - w$ is equal to 5 when $v = -7$ and $w = -12$.
Returning to the equivalent equations $x^2 + (m-7)x + 84 = 0$ and $x^2 - (v+w)x + vw = 0$, we have $m - 7 = -(v + w)$, and so $m = 7 - (v + w)$.
When $v = 12$ and $w = 7$, we get $m = 7 - (12 + 7) = -12$.
When $v = -7$ and $w = -12$, we get $m = 7 - (-7 - 12) = 26$.
The two possible values of m for which the area enclosed by the line and the parabola is as small as possible are -12 and 26.

(c) Suppose the x-coordinates of T and U are the integers t and u respectively, with $t > u$. Then t and u are the solutions of the equation $x^2 + (g + h)x + 9 = -x^2 + gx + h$, which when simplified is $2x^2 + hx + 9 - h = 0$.
Therefore, the equation $2x^2 + hx + 9 - h = 0$ is equivalent to $(x - t)(x - u) = 0$, and so $x^2 + \dfrac{h}{2}x + \dfrac{9-h}{2} = 0$ is equivalent to $x^2 - (t + u)x + tu = 0$.
Equating the constant terms, we get $tu = \dfrac{9-h}{2}$ (equation (1)), and equating the coefficients of the linear terms, we get $-(t + u) = \dfrac{h}{2}$ or $t + u = -\dfrac{h}{2}$ (equation (2)).
Consider the line passing through points T and U, as shown.



The area enclosed by this line and the parabola with equation $y = -x^2 + gx + h$ is equal to $\dfrac{(t-u)^3}{6}$.
Similarly, the area enclosed by this line and the parabola with equation $y = x^2 + (g + h)x + 9$ is also equal to $\dfrac{(t-u)^3}{6}$.
Thus, the area enclosed by the parabolas is equal to $2 \times \dfrac{(t-u)^3}{6} = \dfrac{(t-u)^3}{3}$.
The area enclosed by the parabolas is $\dfrac{3087}{8}$, and so

$$\begin{aligned}
\frac{(t-u)^3}{3} &= \frac{3087}{8} \\
(t-u)^3 &= \frac{9261}{8} \\
t - u &= \sqrt[3]{\frac{9261}{8}} \\
t &= u + \frac{21}{2} \quad \text{(equation (3))}
\end{aligned}$$

Substituting equation (3) into (2), we get $u + \dfrac{21}{2} + u = -\dfrac{h}{2}$ or $h = -4u - 21$.
Substituting $h = -4u - 21$ and equation (3) into equation (1) and solving, we get

$$\begin{aligned}
tu &= \frac{9-h}{2} \\
\left(u + \frac{21}{2}\right)u &= \frac{9-(-4u-21)}{2} \\
(2u+21)u &= 9 + 4u + 21 \\
2u^2 + 17u - 30 &= 0 \\
(2u-3)(u+10) &= 0
\end{aligned}$$

and so the solutions are $u = \dfrac{3}{2}$ and $u = -10$.
When $u = \dfrac{3}{2}$, $h = -4 \cdot \dfrac{3}{2} - 21 = -27$ and when $u = -10$, $h = -4(-10) - 21 = 19$.
The possible values of h for which the area enclosed by the parabolas is $\frac{3087}{8}$ are -27 and 19.

4. (a) The third term of the sequence is $45 \cdot \frac{3}{5} = 27$ and the first term is $45 \div \frac{3}{5} = 45 \cdot \frac{5}{3} = 75$. The sum of the first three terms of the sequence is $75 + 45 + 27 = 147$.

(b) The common ratio, r, of the geometric sequence $x, 12, y$ is $r = \dfrac{12}{x}$, and also $r = \dfrac{y}{12}$.
Thus $\dfrac{y}{12} = \dfrac{12}{x}$ or $xy = 144$.
Substituting $y = 25 - x$ into $xy = 144$ and solving, we get

$$\begin{aligned} x(25-x) &= 144 \\ 25x - x^2 &= 144 \\ x^2 - 25x + 144 &= 0 \\ (x-9)(x-16) &= 0 \end{aligned}$$

and so $x = 9$ and $x = 16$. When $x = 9$, $y = 25 - 9 = 16$ and when $x = 16$, $y = 25 - 16 = 9$.
The possible pairs of positive integers (x, y) are $(9, 16)$ and $(16, 9)$.

(c) Suppose the geometric sequence has common ratio r, so that $b = ar$, $c = ar^2$, and $d = ar^3$.
Since $a + b + c + d = 65$, then

$$\begin{aligned} a + ar + ar^2 + ar^3 &= 65 \\ a(1 + r + r^2 + r^3) &= 65 \\ a((1+r) + r^2(1+r)) &= 65 \\ a(1+r)(1+r^2) &= 65 \end{aligned}$$

Since $r = \dfrac{b}{a}$ and both a and b are non-zero integers, then r is a rational number.
Suppose that $r = \dfrac{m}{n}$ for some non-zero integers m and n having no common factors (that is, $\gcd(m, n) = 1$).
Then $a(1+r)(1+r^2) = 65$ becomes

$$\begin{aligned} a\left(1 + \frac{m}{n}\right)\left(1 + \left(\frac{m}{n}\right)^2\right) &= 65 \\ a\left(\frac{m+n}{n}\right)\left(\frac{m^2+n^2}{n^2}\right) &= 65 \\ \frac{a}{n^3}(m+n)(m^2+n^2) &= 65 \end{aligned}$$

Since d is an integer, and $d = ar^3 = a \cdot \dfrac{m^3}{n^3}$, then $a \cdot \dfrac{m^3}{n^3}$ is an integer.
Since $\gcd(m, n) = 1$, then m^3 and n^3 have no common divisors. We can conclude that n^3 divides a which means that $\dfrac{a}{n^3}$ is an integer.
Suppose $p = \dfrac{a}{n^3}$, then $p(m+n)(m^2+n^2) = 65$ for integers m, n, p.
Each of p, $m+n$ and m^2+n^2 is an integer, and thus a divisor of 65.
The divisors of 65 are $\pm 1, \pm 5, \pm 13$, and ± 65.
Since m and n are non-zero integers, then $m^2 + n^2 \geq 2$ and $m^2 + n^2 \geq m + n$.
We also note that the factors p and $m+n$ are either both positive or they are both negative since $m^2 + n^2$ is positive for all values of m and n. This leads us to consider 3 possible cases: $m^2 + n^2 = 65$, $m^2 + n^2 = 13$, and $m^2 + n^2 = 5$.

Case 1: $m^2 + n^2 = 65$

If $m^2 + n^2 = 65$, then $m + n = 1$ and $p = 1$ or $m + n = -1$ and $p = -1$.
If $m^2 + n^2 = 65$, the integer solutions are $(m, n) = (\pm 1, \pm 8)$ and $(\pm 4, \pm 7)$ and $(\pm 8, \pm 1)$ and $(\pm 7, \pm 4)$. For each of these, $m+n \neq \pm 1$, so there are no solutions when $m^2+n^2 = 65$.

Case 2: $m^2 + n^2 = 13$

When $m^2 + n^2 = 13$, the integer solutions are $(m, n) = (\pm 3, \pm 2)$ and $(\pm 2, \pm 3)$.
This gives the following 8 possibilities:

(i) $(m, n) = (3, 2)$, so $m + n = 5$ and $p = 1 = \dfrac{a}{n^3}$, and so $a = 1 \cdot 2^3 = 8$.
In this case, $r = \dfrac{m}{n} = \dfrac{3}{2}$, and so the quadruple $(a, b, c, d) = (8, 12, 18, 27)$ satisfies the given conditions.

(ii) $(m, n) = (-3, -2)$, so $m + n = -5$ and $p = -1 = \dfrac{a}{n^3}$, and so $a = -1 \cdot (-2)^3 = 8$.
In this case, $r = \dfrac{m}{n} = \dfrac{3}{2}$, and so we get the same quadruple as in (i).

(iii) $(m, n) = (3, -2)$, so $m + n = 1$ and $p = 5 = \dfrac{a}{n^3}$, and so $a = 5 \cdot (-2)^3 = -40$.
In this case, $r = \dfrac{m}{n} = -\dfrac{3}{2}$, and so we get $(a, b, c, d) = (-40, 60, -90, 135)$.

(iv) $(m, n) = (-3, 2)$, so $m + n = -1$ and $p = -5 = \dfrac{a}{n^3}$, and so $a = -5 \cdot (2)^3 = -40$.
In this case, $r = -\dfrac{3}{2}$, and so we get the same quadruple as in (iii).

(v) $(m, n) = (2, 3)$, so $m + n = 5$ and $p = 1$, and so $a = 1 \cdot 3^3 = 27$.
In this case, $r = \dfrac{2}{3}$, and so we get $(a, b, c, d) = (27, 18, 12, 8)$.

(vi) $(m, n) = (-2, -3)$ gives the same quadruple as in (v).

(vii) $(m, n) = (-2, 3)$, so $m + n = 1$ and $p = 5$, and so $a = 5 \cdot 3^3 = 135$.
In this case, $r = \dfrac{m}{n} = -\dfrac{2}{3}$, and so we get $(a, b, c, d) = (135, -90, 60, -40)$.

(viii) $(m, n) = (2, -3)$ gives the same quadruple as in (vii).

Case 3: $m^2 + n^2 = 5$

When $m^2 + n^2 = 5$, the integer solutions are $(m, n) = (\pm 2, \pm 1)$ and $(\pm 1, \pm 2)$.
Since $m^2 + n^2 \geq m + n$, then $m + n = 1$ and $p = 13$, or $m + n = -1$ and $p = -13$.
This gives the following 4 possibilities:

(i) $(m, n) = (2, -1)$, so $m + n = 1$ and $p = 13 = \dfrac{a}{n^3}$, and so $a = 13 \cdot (-1)^3 = -13$.
In this case, $r = -2$, and so we get $(a, b, c, d) = (-13, 26, -52, 104)$.

(ii) $(m, n) = (-2, 1)$ gives the same quadruple as in (i).

(iii) $(m, n) = (-1, 2)$, so $m + n = 1$ and $p = 13$, and so $a = 13 \cdot 2^3 = 104$.
In this case, $r = -\dfrac{1}{2}$, and so we get $(a, b, c, d) = (104, -52, 26, -13)$.

(iv) $(m, n) = (1, -2)$ gives the same quadruple as in (iii).

The quadruples of integers (a, b, c, d) so that a, b, c, d is a geometric sequence and $a+b+c+d = 65$ are: $(8, 12, 18, 27)$, $(27, 18, 12, 8)$, $(-40, 60, -90, 135)$, $(135, -90, 60, -40)$, $(-13, 26, -52, 104)$, and $(104, -52, 26, -13)$. We can verify that each of these quadruples is a geometric sequence with a sum of 65.