The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2025 Gauss Contests*

**(Grades 7 and 8)**

**Wednesday, May 14, 2025**
**(in North America and South America)**

**Thursday, May 15, 2025**
**(outside of North America and South America)**

*Solutions*

**Grade 7**

1. If each of 12 friends gives \$5, in total they give $12 \times \$5 = \$60$ to the charity.

   Answer: (C)

2. The regular hexagon is divided into 6 triangles having equal area. Since 1 of these 6 triangles is shaded, then $\frac{1}{6}$ of the area of the hexagon is shaded.

   Answer: (E)

3. Reading from the graph, Dae ate 6 apples, Joe ate 3, Etta ate 5, Susie ate 4, and Vinh ate 1. Thus, Dae ate the greatest number of apples.
   (We may have simply noted that the person who ate the greatest number of apples is the person with the 'highest bar'.)

   Answer: (A)

4. The equal-arm scale shows that 2 squares have the same mass as 4 circles.
   If we split the 2 squares and the 4 circles each into two equal groups, then 1 square has the same mass as 2 circles.

   Answer: (B)

5. A square with side length 8 has area $8 \times 8 = 64$. Since $8 \times 2 = 16$, $2 \times (8 + 8) = 2 \times 16 = 32$, $4 \times 8 = 32$, and $8 \times 8 \times 8 \times 8 = 4096$, then $8 \times 8$ is the only expression given that is equal to the area of the square.

   Answer: (C)

6. Since $\angle PQR$ is a straight angle, its measure is $180^\circ$. Thus, the angles with measures $130^\circ$ and $x^\circ$ add to $180^\circ$, or $130 + x = 180$ and so $x = 180 - 130 = 50$.

   Answer: (C)

7. In a list of numbers having exactly one mode, the mode is the number that appears in the list the greatest number of times. Since the mode of the given list is 8, and the list has exactly one mode, then 8 must appear more times than each of the other numbers in the list.
   If $n = 8$, then 8 appears in the list three times, which is more than any other number in the list, and so the list has exactly one mode, which is 8, as required.
   If the value of $n$ is 15 or 3, then the list has three numbers that each appear twice, and thus the list does not have exactly one mode.
   If $n = 9$, then 9 appears more times than any other number and so the mode is 9.
   If $n = 10$, then both 8 and 9 each appear twice (which is more than any other number in the list), and thus the list does not have exactly one mode.

   Answer: (D)

8. *Solution 1*
   To measure 1 cup of flour, Sam fills his $\frac{1}{2}$ cup container 2 times (since $\frac{1}{2} \times 2 = 1$).
   To measure 2 cups of flour, Sam fills his $\frac{1}{2}$ cup container $2 \times 2 = 4$ times.
   To measure $2\frac{1}{2}$ cups of flour, Sam fills his $\frac{1}{2}$ cup container $4 + 1 = 5$ times.

   *Solution 2*
   Since $2\frac{1}{2}$ cups is equal to $\frac{5}{2}$ cups, and $5 \times \frac{1}{2} = \frac{5}{2}$, then Sam fills his $\frac{1}{2}$ cup container 5 times.

   Answer: (E)

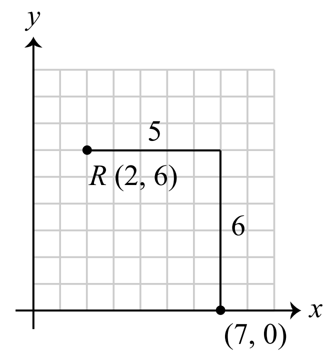
9. There are 7 days in a week. If June 1 is a Tuesday, then by moving successively 7 days later in the month, we get June 8, 15, 22, and 29 are also Tuesdays. Thus, June 30 is a Wednesday.

ANSWER: (C)

10. When viewed from the opposite side of the window, text appears reflected horizontally.
That is, PUG FOR SALE appears as Ǝ⅃A Ƨ ЯOꟻ ӘUꟼ. The letters that look the same when viewed from both sides of the window are A, O and U and so there are 3 such letters.

ANSWER: (A)

11. The $x$-coordinate of $(2,6)$ is 5 less than the $x$-coordinate of $(7,0)$, and so $R$ must be translated right 5.
The $y$-coordinate of $(2,6)$ is 6 more than the $y$-coordinate of $(7,0)$, and so $R$ must be translated down 6.
Thus, the translation right 5, down 6 will move $R(2,6)$ to $(7,0)$.




ANSWER: (E)

12. A train stops at Waterloo Station at 6:25 a.m. and every 3 minutes thereafter, and so a train stops only at times which differ from 6:25 a.m. by a multiple of 3. Of the given answers, a train stops at the station at 6:28 a.m. (difference is 3 minutes), at 6:40 a.m. (difference is 15 minutes), and at 6:55 a.m. (difference is 30 minutes).
A bus stops at Waterloo Station at 6:25 a.m. and every 5 minutes thereafter, and so a bus stops only at times which differ from 6:25 a.m. by a multiple of 5. Of the given answers, a bus stops at the station at 6:30 a.m. (difference is 5 minutes), at 6:40 a.m. (difference is 15 minutes), and at 6:55 a.m. (difference is 30 minutes).
Thus, the next time that both a train and a bus stop at Waterloo Station at the same time is 6:40 a.m.
Note: The lowest common multiple of both 3 and 5 is 15. This tells us that a train and a bus will stop at the station every 15 minutes after 6:25 a.m. This confirms that 6:40 a.m. is the next time that both a train and a bus will stop at the station at the same time.

ANSWER: (D)

13. The pattern repeats in blocks of the 4 numbers 2, 0, 2, 5. Since $50 = 12 \times 4 + 2$, then the 50 numbers in the pattern contain 12 complete blocks of 2, 0, 2, 5 followed by the first 2 numbers in the block, 2, 0.
The number 5 appears once in each of the 12 blocks and does not appear as the 49th or 50th number. Thus, the number 5 appears 12 times.

ANSWER: (C)

14. Since $\frac{28}{32} + \frac{1}{\square} = 1$ and $\frac{28}{32} + \frac{4}{32} = \frac{32}{32} = 1$, then $\frac{1}{\square} = \frac{4}{32}$.
Reducing $\frac{4}{32}$ to lowest terms, we get $\frac{4}{32} = \frac{1}{8}$, and so the number that goes in the box is 8.

ANSWER: (E)

15. In the grid shown, the first column and first row show the possible numbers that may appear on the top faces of the two dice. The inside of the grid shows the sum of those corresponding two numbers. Of the 36 possible combinations of rolls, a sum of 7 can happen 6 different ways, a sum of 8 can happen 5 different ways, a sum of 9 can happen 4 different ways, a sum of 10 can happen 3 different ways, and a sum of 11 can happen 2 different ways. Thus, of the given sums, 11 is the least likely to occur.

| + | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Answer: (E)

16. For the greatest possible result, we want the sum of the first two 2-digit numbers to be as large as possible, and the third 2-digit number to be as small as possible.
Since the tens digit contributes more to the value of a number than the ones digit, we choose the two largest digits, 7 and 8, as the tens digits of the first two 2-digit numbers, and the smallest of the given digits, 1, as the tens digit of the third 2-digit number.
Of the remaining digits, 3, 6, and 2, we choose the two largest digits, 3 and 6, as the ones digits of the first two 2-digit numbers, and 2 as the ones digit of the third 2-digit number, which becomes 12.
Since $73 + 86 = 76 + 83$, it does not matter how we pair the tens digits with the ones digits for the first two 2-digit numbers.
Thus, the greatest possible result is $73 + 86 - 12 = 147$.

Answer: (C)

17. *Solution 1*
We begin by recognizing that written as a fraction in lowest terms, 60% is equal to $\frac{3}{5}$.
Thus, if Savanah tossed 3 tails out of 5 tosses, then 60% of the tosses that she made were tails. Since the result of her last toss was tails, then following Savanah's second last toss she had tossed 2 tails out of 4 tosses, meaning that 50% of the tosses that she made to that point were tails, as required.
Therefore, Savanah made 5 tosses in total.

*Solution 2*
We proceed to work backward from the answers given.
If Savanah made 3 tosses, and 60% of those tosses were tails, then she tossed $3 \times \frac{60}{100} = \frac{180}{100} = 1.8$ tails. She must toss a whole number of tails, and so this is not possible.
If Savanah made 9 tosses, then she tossed $9 \times \frac{60}{100} = \frac{540}{100} = 5.4$ tails. Again, this is not possible.
If Savanah made 8 tosses, then her second last toss was her 7th toss. However, it is not possible for Savanah to toss 50% tails in 7 tosses (since one half of 7 is not a whole number). (We could have instead shown that 60% of 8 is also not an integer as we did in the first two cases.)
Similarly, Savanah could not have made 10 tosses since one half of 9 tosses is not an integer.
By process of elimination, the answer must be 5 tosses. We can confirm that if she tosses 2 tails in her first 4 tosses, then 50% of her tosses are tails, and if her final toss is a tail, then she has tossed 3 tails out of 5 tosses, which is 60% tails, as required.

Answer: (D)

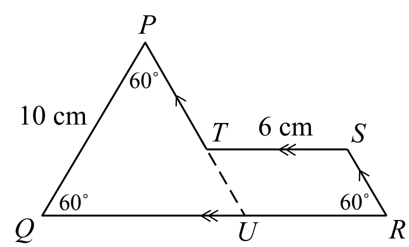
18. The sum of the four angle measurements in a quadrilateral is $360^\circ$.
The sum of the five angle measurements given is $62^\circ + 85^\circ + 99^\circ + 108^\circ + 114^\circ = 468^\circ$.
The difference, $468^\circ - 360^\circ = 108^\circ$, is the measure of the angle that is not in the quadrilateral.
(We may confirm that $62^\circ + 85^\circ + 99^\circ + 114^\circ = 360^\circ$.)

Answer: (D)

19. Each integer is either an odd number or it is an even number.
Thus, each of the 10 students will check off exactly one box from the first pair of boxes.
Similarly, every integer greater than 1 is either a prime number or it is a composite number, and so each of the 10 students will check off exactly one box from the second pair of boxes.
Thus, every student checks off exactly 2 of the first 4 boxes.
This means that every student checks off exactly 2 of the 5 boxes if their card *is not* numbered with a perfect square, and they check off exactly 3 of the 5 boxes if their card *is* numbered with a perfect square.
The card numbered 16 is the only card numbered with a perfect square, and so each of the remaining $10 - 1 = 9$ students have a card that is not numbered with a perfect square.
Thus, there are 9 students that check off exactly two boxes.

Answer: (B)

20. We begin by extending $PT$ to point $U$ on $QR$, as shown.
Since $PT$ is parallel to $SR$, then $TU$ is parallel to $SR$.
Similarly, since $TS$ is parallel to $QR$, then $TS$ is parallel to $UR$.
Since $TU$ is parallel to $SR$ and $TS$ is parallel to $UR$, then $TURS$ is a parallelogram, and so $TU = SR$ and $UR = TS = 6$ cm.
In $\triangle PQU$, the sum of the measures of the three angles is $180^\circ$, and so $\angle PUQ = 180^\circ - 60^\circ - 60^\circ = 60^\circ$.




Thus, $\triangle PQU$ is an equilateral triangle, and so $QU = PU = PQ = 10$ cm.
The perimeter of $PQRST$ is

$$\begin{aligned} PQ + QR + SR + TS + PT &= PQ + QU + UR + SR + TS + PT \quad (\text{since } QR = QU + UR) \\ &= PQ + QU + UR + TU + TS + PT \quad (\text{since } SR = TU) \\ &= PQ + QU + UR + PT + TU + TS \quad (\text{reordering some sides}) \\ &= PQ + QU + UR + PU + TS \quad (\text{since } PT + TU = PU) \\ &= 10 \text{ cm} + 10 \text{ cm} + 6 \text{ cm} + 10 \text{ cm} + 6 \text{ cm} \\ &= 42 \text{ cm} \end{aligned}$$

Answer: (A)

21. A circle with radius 1 cm has area $\pi \times (1 \text{ cm})^2 = \pi \text{ cm}^2$.
A circle with radius 5 cm has area $\pi \times (5 \text{ cm})^2 = 25\pi \text{ cm}^2$.
A circle with radius $x$ cm has area $\pi \times (x \text{ cm})^2 = x^2\pi \text{ cm}^2$.
Since the mean area of the three circles is $30\pi \text{ cm}^2$, then the sum of the areas of the three circles is $3 \times 30\pi \text{ cm}^2 = 90\pi \text{ cm}^2$.
Since $\pi + 25\pi + x^2\pi = 26\pi + x^2\pi$, we get $26\pi + x^2\pi = 90\pi$ or $x^2\pi = 64\pi$ or $x^2 = 64$, and so $x = 8$ (since $x > 0$).

Answer: (D)

22. *Solution 1*

The probability that at least one colour is not used is equal to 1 minus the probability that all three colours are used.

Each of the three doors can be painted one of three different colours, and so there are $3 \times 3 \times 3 = 27$ different ways to paint the doors with no restrictions.

If all 3 colours are used, then the first door can be painted 3 different colours, the second door can be painted 2 different colours, and the third door must be painted the 1 remaining colour.

The probability that all three colours are used is thus $\frac{3 \times 2 \times 1}{27} = \frac{6}{27}$, and so the probability that at least one colour is not used is $1 - \frac{6}{27} = \frac{27}{27} - \frac{6}{27} = \frac{21}{27}$, which is equal to $\frac{7}{9}$.

*Solution 2*

The probability that at least one colour is not used is equivalent to the probability that at most two different colours are used.

Thus, the probability that at least one colour is not used is equal to the sum of the probability that exactly one colour is used and the probability that exactly two different colours are used.

We begin by determining the probability that exactly one colour is used.

Each of the three doors can be painted one of three different colours, and so there are $3 \times 3 \times 3 = 27$ different ways to paint the doors with no restrictions.

If exactly one colour is used, there are 3 choices for the colour and each door must be painted with that colour, and so the probability that exactly one colour is used is $\frac{3}{27}$.

If exactly two colours are used, then two doors are painted the same colour and the remaining door is painted a different colour.

There are 3 choices for the first colour and then 2 choices for the second colour, and thus $3 \times 2 = 6$ ways to choose the two colours. Once the two colours are chosen, the doors can be painted with these colours in 3 different ways. For example, if the colour chosen for two of the doors is black (B) and the colour for the remaining door is white (W), the doors can be painted BBW, BWB or WBB.

Therefore, the number of ways to paint the doors with exactly two different colours is $6 \times 3 = 18$, and so the probability that exactly two different colours are used is $\frac{18}{27}$.

Finally, the probability that at least one colour is not used is $\frac{3}{27} + \frac{18}{27} = \frac{21}{27} = \frac{7}{9}$.

ANSWER: (A)

23. *Solution 1*

We are being asked to find the number of multiples of 18 between 111 000 and 111 999 inclusive.

Since $18 \times 6166 = 110\,988$ and $18 \times 6167 = 111\,006$, then the smallest multiple of 18 greater than or equal to 111 000 is 111 006. (We can find the numbers 6166 and 6167 by dividing 111 000 by 18 and rounding the result both down and up to the nearest integer.)

Since $18 \times 6222 = 111\,996$ and $18 \times 6223 = 112\,014$, then the largest multiple of 18 less than or equal to 111 999 is 111 996. (We can similarly find the numbers 6222 and 6223 by dividing 111 999 by 18 and rounding the result both down and up to the nearest integer.)

All multiples of 18 from $18 \times 6167$ to $18 \times 6222$ inclusive satisfy the given conditions, and so there are $6222 - 6167 + 1 = 56$ different possibilities for $N$.

*Solution 2*

Since $N = 111\,abc$ is divisible by 18, then $N$ is divisible by both 2 and 9 since 2 and 9 have no factors in common and $2 \times 9 = 18$.

Since $N$ is divisible by 2, then $N$ is even and so its units digit, $c$, must equal 0, 2, 4, 6, or 8.
An integer is divisible by 9 exactly when the sum of its digits is divisible by 9.
Thus, the sum of the digits of $N$, which is equal to $1+1+1+a+b+c = a+b+c+3$, must be a multiple of 9.
The smallest possible value of $a+b+c+3$ is 3 (when $a=b=c=0$) and its largest possible value is 30 (when $a=b=c=9$).
The multiples of 9 in this range are 9, 18 and 27, and so $a+b+c=6$ or $a+b+c=15$ or $a+b+c=24$.
To summarize to this point, $N = 111\,abc$ is divisible by 18 exactly when $c$ is equal to 0, 2, 4, 6, or 8, and $a+b+c=6$ or $a+b+c=15$ or $a+b+c=24$.
We proceed by setting $c$ equal to each of the possible even digits and determining the number of possible values for $a$ and $b$.

When $c=0$, we get $a+b=6$ or $a+b=15$ or $a+b=24$.
When $a+b=6$, the possible values for the digits $a$ and $b$ (written as ordered pairs $(a,b)$) are $(0,6)$, $(1,5)$, $(2,4)$, $(3,3)$, $(4,2)$, $(5,1)$, and $(6,0)$. (Alternately, when $a+b=6$, $a$ can equal any integer from 0 to 6 inclusive and then $b=6-a$.)
Thus, there are 7 ordered pairs of digits $(a,b)$ when $c=0$ and $a+b=6$, and so there are 7 possible values of $N$.
When $a+b=15$, the possible pairs of digits $(a,b)$ are $(6,9)$, $(7,8)$, $(8,7)$, and $(9,6)$, and so there are 4 possible values of $N$.
When $a+b=24$, there are no possible pairs of digits $(a,b)$ since $a+b$ is at most $9+9=18$.

When $c=2$, we get $a+b=6-2=4$ or $a+b=15-2=13$ or $a+b=24-2=22$.
We continue to count the number of pairs of digits $(a,b)$ given each of the possible values of $c$ and summarize those results in the tables below.

| $c=0$ | $a+b=6$ | $a+b=15$ | $a+b=24$ | Number of values of $N$ |
|---|---|---|---|---|
| Number of pairs $(a,b)$ | 7 | 4 | 0 | 11 |

| $c=2$ | $a+b=4$ | $a+b=13$ | $a+b=22$ | Number of values of $N$ |
|---|---|---|---|---|
| Number of pairs $(a,b)$ | 5 | 6 | 0 | 11 |

| $c=4$ | $a+b=2$ | $a+b=11$ | $a+b=20$ | Number of values of $N$ |
|---|---|---|---|---|
| Number of pairs $(a,b)$ | 3 | 8 | 0 | 11 |

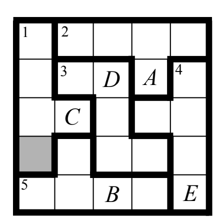
| $c=6$ | $a+b=0$ | $a+b=9$ | $a+b=18$ | Number of values of $N$ |
|---|---|---|---|---|
| Number of pairs $(a,b)$ | 1 | 10 | 1 | 12 |

| $c=8$ | $a+b=-2$ | $a+b=7$ | $a+b=16$ | Number of values of $N$ |
|---|---|---|---|---|
| Number of pairs $(a,b)$ | 0 | 8 | 3 | 11 |

Therefore, the number of possibilities for $N$ is $11+11+11+12+11=56$.

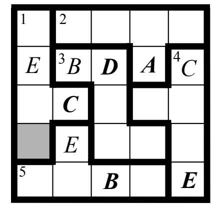
ANSWER: (C)

24. We begin by naming shapes shown by the thick lines 1, 2, 3, 4, and 5, as shown in the diagram.



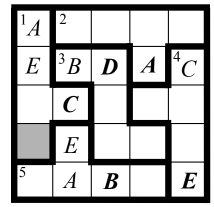

We also adopt the notation [row number, column number] to refer to the contents of specific squares in the diagram.
For example, we are given that [2, 3] is $D$ (this is in shape 3), and [2, 4] is $A$ (this is in shape 2).

It is important to note that the solution that follows is one of many different ways to arrive at the final answer.



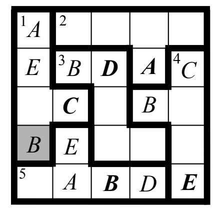
Since $[3,2]$ is $C$, then $[2,1]$ cannot be $C$ (shape 1 already has $C$), and $[2,2]$ cannot be $C$ (column 2 already has $C$), and thus the $C$ in row 2 must be $[2,5]$.
Since row 5 already has an $E$, then the $E$ in shape 5 cannot be in row 5, and thus $[4,2]$ is $E$.

Row 2 is missing $B$ and $E$ and since column 2 has an $E$, then $[2,1]$ is $E$ and $[2,2]$ is $B$. The letters determined to this point are included in the diagram above.



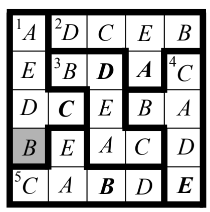
Since shape 2 already contains $A$ and row 1 is missing $A$, then $[1,1]$ is $A$.
Also, column 4 already contains $A$ and so $[5,4]$ is not $A$ which means that $[5,2]$ is $A$ (since shape 5 is missing $A$).
The letters determined to this point are included in the diagram to the right.

Shape 1 must contain $D$ somewhere in column 1 and so $[5,1]$ is not $D$. Since shape 5 is missing $D$, then $[5,4]$ is $D$.



Shape 4 is missing $A$, $B$ and $D$. Since column 4 already contains $A$ and $D$, then $[3,4]$ cannot be $A$ or $D$, and thus $[3,4]$ is $B$.
Finally, shape 1 is missing $B$ and $D$. Since row 3 contains $B$, then $[3,1]$ cannot be $B$, and so the letter in the shaded square, $[4,1]$, is $B$ as shown.

The completed diagram is included below. Given the initial 5 letters and their locations, there is exactly one way to complete this diagram.



Answer: (B)

25. We adopt the notation [row number, column number] to be equal to the contents of a specific square in the grid.
Since $[1,1]$ is 1 and adjacent numbers increase by a fixed integer $a > 0$ moving left to right within each row, then $[1,2]$ is $1+a$, $[1,3]$ is $1+2a$, and so on to the end of the row where $[1,8]$ is $1+7a$.
Similarly, adjacent numbers increase by a fixed integer $b > 0$ moving top to bottom within each column, and so the contents of each square in row 2 is $b$ greater than the adjacent square in row 1.
That is, $[2,1]$ is $1+b$, $[2,2]$ is $1+a+b$, $[2,3]$ is $1+2a+b$, and so on to the end of the row where $[2,8]$ is $1+7a+b$.
Continuing in this way in row 3, we get $[3,1]$ is $1+2b$, $[3,2]$ is $1+a+2b$, $[3,3]$ is $1+2a+2b$, and so on to $[3,8]$ which is equal to $1+7a+2b$.
We continue this pattern in the table below, and include each of the entries in column 5 since the focus of the question is on this column.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $1$ | $1+a$ | $1+2a$ | $1+3a$ | $1+4a$ | $1+5a$ | $1+6a$ | $1+7a$ |
| $1+b$ | $1+a+b$ | $1+2a+b$ | $1+3a+b$ | $1+4a+b$ | $1+5a+b$ | $1+6a+b$ | $1+7a+b$ |
| $1+2b$ | $1+a+2b$ | $1+2a+2b$ | $1+3a+2b$ | $1+4a+2b$ | $1+5a+2b$ | $1+6a+2b$ | $1+7a+2b$ |
| $1+3b$ | $1+a+3b$ | $\vdots$ | $\vdots$ | $1+4a+3b$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $1+4b$ | $1+a+4b$ | | | $1+4a+4b$ | | | |
| $1+5b$ | $1+a+5b$ | | | $1+4a+5b$ | | | |
| $1+6b$ | $1+a+6b$ | | | $1+4a+6b$ | | | |
| $1+7b$ | $1+a+7b$ | | | $1+4a+7b$ | | | $1+7a+7b$ |

We are given that the number in the bottom right corner of the grid is less than 75, and so $1 + 7a + 7b < 75$ or $7a + 7b < 74$.
Dividing each term of this inequality by 7, we get $\frac{7a}{7} + \frac{7b}{7} < \frac{74}{7}$ or $a + b < \frac{74}{7}$ and since $a$ and $b$ are positive integers, then $a + b \leq 10$.
Since $a + b \leq 10$, then by multiplying each term by 4, we get $4a + 4b \leq 40$.
In the table above, $[5, 5]$ is $1 + 4a + 4b$. Since $4a + 4b \leq 40$, then $1 + 4a + 4b \leq 41$ and so $[5, 5]$ cannot equal 45.
Further, since $b$ is a positive integer, then each of the entries in column 5 above row 5 is less than $1 + 4a + 4b$ and thus cannot equal 45.
Thus if 45 appears in column 5 of this grid, then it can only appear in rows 6, 7 and 8.

We begin by noting that $a$ and $b$ are positive integers, and since $a + b \leq 10$, then $a \leq 9$ and $b \leq 9$.
If $[6, 5]$ is 45, then $1 + 4a + 5b = 45$ or $4a + 5b = 44$.
If $a = 1$, then $4 \times 1 + 5b = 44$ or $5b = 40$, and so $b = 8$.
We confirm that when $a = 1$ and $b = 8$, then $a + b \leq 10$ and so the number appearing in the bottom right corner of the grid is less than 75.
Thus, in the arithmetic grid with $a = 1$ and $b = 8$, $[6, 5]$ is 45.
If $a = 2$, then $4 \times 2 + 5b = 44$ or $5b = 36$ or $b = \frac{36}{5}$ (which is not an integer), and so there is no such arithmetic grid in which $[6, 5]$ is 45 when $a = 2$.
We could continue substituting the remaining values of $a$ from 3 to 9 to determine for which values of $a$, $4a + 5b = 44$ and $a + b \leq 10$ and $b$ is a positive integer.
Alternately, we might notice that $4a$ is a multiple of 4, as is 44.
Thus, if $4a + 5b = 44$ then $5b$ must also be a multiple of 4 and so $b$ must be a multiple of 4.
The only remaining positive integer $b$ for which $b \leq 9$ and $b$ is a multiple of 4 is $b = 4$.
When $b = 4$, we get $4a + 5 \times 4 = 44$ or $4a = 24$ and so $a = 6$.
We again confirm that when $a = 6$ and $b = 4$, then $a + b \leq 10$ and so the number appearing in the bottom right corner of the grid is less than 75.
In the arithmetic grid with $a = 6$ and $b = 4$, $[6, 5]$ is 45.
Thus, there are exactly 2 such grids in which 45 appears in row 6, column 5.

If $[7, 5]$ is 45, then $1 + 4a + 6b = 45$ or $4a + 6b = 44$ or $2a + 3b = 22$.
In this case, we similarly notice that $2a$ is a multiple of 2, as is 22.
Thus, if $2a + 3b = 22$ then $3b$ must also be a multiple of 2 and so $b$ must be a multiple of 2.
We proceed by checking values of $b$ which are equal to positive even integers.
When $b = 2$, we get $2a + 3 \times 2 = 22$ or $2a = 16$ and so $a = 8$.
In this case, $a + b \leq 10$ and so the number appearing in the bottom right corner is less than 75.
In the arithmetic grid with $a = 8$ and $b = 2$, $[7, 5]$ is 45.
When $b = 4$, we get $2a + 3 \times 4 = 22$ or $2a = 10$ and so $a = 5$.
In this case, $a + b \leq 10$.

In the arithmetic grid with $a = 5$ and $b = 4$, $[7, 5]$ is 45.
When $b = 6$, we get $2a + 3 \times 6 = 22$ or $2a = 4$ and so $a = 2$.
In this case, $a + b \leq 10$.
In the arithmetic grid with $a = 2$ and $b = 6$, $[7, 5]$ is 45.
When $b = 8$, we get $2a + 3 \times 8 = 22$ which is not possible since $a > 0$.
Thus, there are exactly 3 such grids in which 45 appears in row 7, column 5.

If $[8, 5]$ is 45, then $1 + 4a + 7b = 45$ or $4a + 7b = 44$.
We notice that $4a$ is a multiple of 4, as is 44.
Thus, if $4a + 7b = 44$ then $7b$ must also be a multiple of 4 and so $b$ must be a multiple of 4.
We proceed by checking the two possible values of $b$, namely $b = 4$ and $b = 8$.
When $b = 4$, we get $4a + 7 \times 4 = 44$ or $4a = 16$ and so $a = 4$.
In this case, $a + b \leq 10$ and so the number appearing in the bottom right corner is less than 75.
In the arithmetic grid with $a = 4$ and $b = 4$, $[8, 5]$ is 45.
When $b = 8$, we get $4a + 7 \times 8 = 44$ which is not possible since $a > 0$.
Thus, there is exactly 1 such grid in which 45 appears in row 8, column 5, and so there are $2 + 3 + 1 = 6$ grids in total.

Answer: (A)

**Grade 8**

1. In each row, exactly 3 of the 6 or one-half of the circles are shaded. Thus, one-half of the 24, that is, 12 circles are shaded. Alternately, we could just count the 12 shaded circles.

   Answer: (B)

2. If each of 4 children shared 36 pieces equally, then each child received $\frac{36}{4} = 9$ pieces.

   Answer: (D)

3. There are 7 days in each week, and so there are $2 \times 7 = 14$ days in two weeks.
   Thus, the date two weeks from May 12 is May 26 (since $12 + 14 = 26$).

   Answer: (D)

4. *Solution 1*
   The time 2 hours after 8:45 a.m. is 10:45 a.m., and the time 45 minutes after 10:45 a.m. is 11:30 a.m.

   *Solution 2*
   Two hours and 45 minutes is 15 minutes less than 3 hours.
   The time 3 hours after 8:45 a.m. is 11:45 a.m., and the time 15 minutes before 11:45 a.m. is 11:30 a.m.

   Answer: (C)

5. If $7x - 3 = 60$, then $7x = 60 + 3$, so $7x = 63$ and $x = \frac{63}{7} = 9$.

   Answer: (A)

6. There are 2 ways to colour the triangle and 3 ways to colour the square, and so there are $2 \times 3 = 6$ ways to colour the figure.
   We can write the 6 colourings of the figure as ordered pairs in which the first colour listed in the pair is the colour of the triangle and the second is the colour of the square. These are: (red, blue), (red, purple), (red, green), (yellow, blue), (yellow, purple), and (yellow, green).

   Answer: (D)

7. When the point $(5, 7)$ is reflected in the $x$-axis, the resulting point is $(5, -7)$.
   When a point is reflected in the $x$-axis, its $x$-coordinate does not change and its $y$-coordinate changes sign (provided that the original point is not on the $x$-axis).

   Answer: (A)

8. Reading from the graph, 5 students voted for Spring, 15 voted for Summer, 5 voted for Fall, and 10 voted for Winter.
   Therefore, (A) Fall and Spring received the same number of votes, (B) Winter received more votes than Spring, (C) 35 students participated in the survey, and (E) 5 students voted for Fall, are all true statements.
   Since 15 students voted for Summer and 15 is less than half of 35, then (D) is the statement that is false.

   Answer: (D)

9. *Solution 1*
The sum of the original five digits is $5 + 2 + 8 + 7 + 9 = 31$.
The largest multiple of 4 that is less than 31 is 28, and 28 is 3 less than 31.
However, 3 is not among the list of five digits, and thus Ruhab cannot erase a 3.
The next largest multiple of 4 that is less than 31 is 24.
Since $31 - 24 = 7$, and 7 does appear in the original list of five digits, then if Ruhab erases the 7, the sum of the remaining four digits is a multiple of 4.
(We may confirm that $5 + 2 + 8 + 9 = 24$.)

*Solution 2*
We may erase each of the five digits one at a time and in each case, determine the sum of the remaining four digits.
Doing so, we get

$$5 + 2 + 8 + 7 = 22;\ 5 + 2 + 8 + 9 = 24;\ 5 + 2 + 7 + 9 = 23;$$

$$5 + 8 + 7 + 9 = 29;\ 2 + 8 + 7 + 9 = 26$$

Of these sums, only 24 is a multiple of 4 and so Ruhab erased the digit 7.

Answer: (D)

10. The perfect squares are $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, and so on.
There are 18 integers between 3 and 20 inclusive.
Of these 18 integers, only $2^2 = 4$, $3^2 = 9$ and $4^2 = 16$ are perfect squares.
(We note that $1^2 = 1$ and $5^2 = 25$ are not among the integers given.)
The probability of randomly selecting a perfect square from the integers given is $\frac{3}{18} = \frac{1}{6}$.

Answer: (C)

11. Since $\frac{28}{32} + \frac{1}{\square} = 1$ and $\frac{28}{32} + \frac{4}{32} = \frac{32}{32} = 1$, then $\frac{1}{\square} = \frac{4}{32}$.
Reducing $\frac{4}{32}$ to lowest terms, we get $\frac{4}{32} = \frac{1}{8}$, and so the number that goes in the box is 8.

Answer: (E)

12. There are 60 minutes in one hour.
Since $3 \times 20$ minutes $= 60$ minutes, and $3 \times 1.5$ km $= 4.5$ km, then Leticia can walk 4.5 km in one hour.
Walking at this same rate, Leticia can walk $4 \times 4.5$ km $= 18$ km in 4 hours.

Answer: (A)

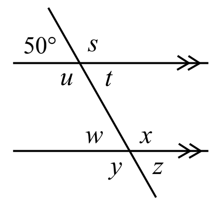
13. Ordered from smallest to largest, the list is

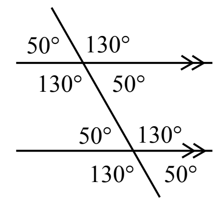
$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6$$

This list contains $1 + 2 + 3 + 4 + 5 + 6 = 21$ integers.
The median is the middle number in an ascending (or descending) list of numbers.
The 11th integer in the list of 21 integers above has 10 integers before it and 10 integers following it, and thus is the median of the list.
The 11th integer and the median of the given list is 5.

Answer: (D)

14. Opposite angles are equal in measure and so $t = 50^\circ$ since it is opposite the angle whose measure is given as $50^\circ$.
By a property of parallel lines, the angles whose measures are $t$ and $x$ form a "C pattern", and are supplementary.
Thus $t + x = 180^\circ$ or $x = 180^\circ - 50^\circ = 130^\circ$.
Since the angle whose measure is $y$ is opposite the angle whose measure is $x$, then $y = x = 130^\circ$.
Since $t + y = 50^\circ + 130^\circ = 180^\circ$, then from the given answers, $t$ and $y$ are the pair of angles whose measures sum to $180^\circ$.
Of the answers given, we confirm that no other pair of angles have measures whose sum is $180^\circ$ by determining the measures of each of the missing angles, as shown.




Answer: (D)

15. *Solution 1*
The mean age of the first three students is 13, and so the sum of their ages is $13 \times 3 = 39$.
The mean age of the four students is 14, and so the sum of their ages is $14 \times 4 = 56$.
The age of the fourth student is the difference between these two sums, which is $56 - 39 = 17$ years old.

*Solution 2*
In an ordered list of 3 consecutive integers, the average of the list is always the middle integer. Can you see why?
Therefore, three students whose ages are consecutive integers and whose mean age is 13 are 12, 13 and 14 years old.
Suppose the fourth student is $x$ years of age.
Since the mean age of the four students is 14, then $\dfrac{12 + 13 + 14 + x}{4} = 14$ or $39 + x = 14 \times 4$, and so $x = 56 - 39 = 17$.
The fourth student is 17 years old.

Answer: (E)

16. There is 1 dog for every bowl of food, and so if there were 6 dogs, then there would be 6 bowls of food.
There are 2 dogs for every bowl of water, and so these same 6 dogs would need $\frac{6}{2} = 3$ bowls of water.
There are 3 dogs for every bowl of treats, and so the 6 dogs would need $\frac{6}{3} = 2$ bowls of treats.
Thus, 6 dogs require 6 bowls of food, 3 bowls of water, and 2 bowls of treats, or $6 + 3 + 2 = 11$ bowls in total.
There are 11 bowls for every 6 dogs, and since there are $77 = 11 \times 7$ bowls, then there are 7 groups of 6 dogs, or $6 \times 7 = 42$ dogs.

Answer: (C)

17. In $\triangle CBD$, $CB = BD$ and so $\angle BCD$ and $\angle BDC$ have equal measures. The sum of the angles in a triangle is $180^\circ$, and since $\angle CBD = 90^\circ$, then $\angle BCD = \angle BDC = 45^\circ$.
(We confirm that $90^\circ + 45^\circ + 45^\circ = 180^\circ$.)
Since $\angle CDE$ is a straight angle, then $\angle BDE + \angle BDC = 180^\circ$ or $\angle BDE = 180^\circ - 45^\circ = 135^\circ$.
In $\triangle BDE$, $BD = DE$ and so $\angle DBE$ and $\angle DEB$ have equal measures. The sum of the angles in a triangle is $180^\circ$, and since $\angle BDE = 135^\circ$, then $\angle DBE = \dfrac{180^\circ - 135^\circ}{2} = 22.5^\circ$.
Finally, since $\angle ABC$ is a straight angle, then $\angle ABE + \angle DBE + \angle DBC = 180^\circ$ or $\angle ABE = 180^\circ - 22.5^\circ - 90^\circ = 67.5^\circ$.

ANSWER: (B)

18. In the grid shown, the first column and first row show the possible numbers that may appear on the top faces of the two dice.
The inside of the grid shows the product of the corresponding two numbers. Of the results, a product of 4 appears 3 times, a product of 6 appears 4 times, a product of 9 appears 1 time, a product of 15 appears 2 times, and a product of 8 appears 2 times. Thus, of the possible products given, 6 is the most likely to occur.

| × | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 4 | 6 | 8 | 10 | 12 |
| 3 | 3 | 6 | 9 | 12 | 15 | 18 |
| 4 | 4 | 8 | 12 | 16 | 20 | 24 |
| 5 | 5 | 10 | 15 | 20 | 25 | 30 |
| 6 | 6 | 12 | 18 | 24 | 30 | 36 |

ANSWER: (B)

19. We begin by determining the prime factorization of 2025.

$$\begin{aligned} 2025 &= 25 \times 81 \\ &= 5 \times 5 \times 9 \times 9 \\ &= 5 \times 5 \times 3 \times 3 \times 3 \times 3 \\ &= 3^4 \times 5^2 \end{aligned}$$

There are exactly 15 positive factors of 2025. These are:

$$1, 3, 3^2, 3^3, 3^4, 5, 5^2, 3 \times 5, 3^2 \times 5, 3^3 \times 5, 3^4 \times 5, 3 \times 5^2, 3^2 \times 5^2, 3^3 \times 5^2, \text{ and } 3^4 \times 5^2$$

We are asked to express 2025 as the product of two positive integers $n$ and $m^2$, where $m^2$ is a perfect square. Of the 15 positive factors, the following are perfect squares:

$$1, 3^2, 3^4, 5^2, 3^2 \times 5^2, \text{ and } 3^4 \times 5^2$$

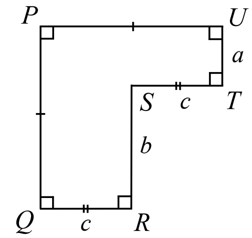
These are all the possible values of $m^2$, and so the values of $m$ are: $1, 3, 5, 3^2, 3 \times 5$, and $3^2 \times 5$.
Thus, the ordered pairs of positive integers $(m, n)$ for which $m^2 \times n = 2025$ are:

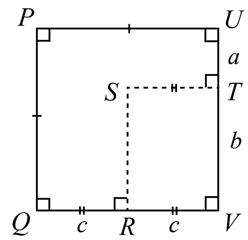
$$(1, 3^4 \times 5^2), (3, 3^2 \times 5^2), (5, 3^4), (3^2, 5^2), (3 \times 5, 3^2), (3^2 \times 5, 1)$$

Therefore, there are 6 such ordered pairs.
It is interesting to note that each of the 6 values of $n$ is also a perfect square. Can you see why this occurs?

ANSWER: (E)

20. In the first diagram shown, we label the vertices of the polygon and the length $ST = c$, since $ST = QR$. Next, we extend $UT$ by a length equal to $SR$, and we extend $QR$ by a length equal to $ST$, as shown in the second diagram. Each of the angles in the polygon is a right angle, and so these two extended line segments are perpendicular to each other and will meet at a point that we label $V$.




That is, $STVR$ is a rectangle with $TV = SR = b$ and $RV = ST = c$.
Each of the following expressions is equal to the perimeter of the original polygon

$$\begin{aligned} & PQ + QR + SR + ST + TU + PU \\ =\ & PQ + QR + ST + SR + TU + PU \text{ (reordering the lengths)} \\ =\ & PQ + QR + RV + TV + TU + PU \text{ (since } RV = ST \text{ and } TV = SR) \\ =\ & PQ + QV + UV + PU \text{ (since } QR + RV = QV \text{ and } TV + TU = UV) \end{aligned}$$

which is the perimeter of $PQVU$.

Each of the angles in $PQVU$ is a right angle, and $PQ = PU$, and thus $PQVU$ is a square.
Since $PQ = UV = UT + TV = a + b$, and $PU = QV = QR + RV = c + c = 2c$, then $a + b = 2c$.
Summarizing, the perimeter of the original polygon is equal to the perimeter of square $PQVU$, and each side length of square $PQVU$ can be expressed as $a + b$ or as $2c$ since $a + b = 2c$.
If each of the 4 side lengths is expressed as $a + b$, then the perimeter of $PQVU$ (and thus the perimeter of the original polygon), is equal to $(a + b) + (a + b) + (a + b) + (a + b) = 4a + 4b$.
If 3 side lengths are expressed as $a + b$ and 1 side length is expressed as $2c$, then the perimeter is $(a + b) + (a + b) + (a + b) + (2c) = 3a + 3b + 2c$.
If 2 side lengths are expressed as $a+b$ and 2 side lengths are expressed as $2c$, then the perimeter is $(a + b) + (a + b) + (2c) + (2c) = 2a + 2b + 4c$.
If 1 side length is expressed as $a + b$ and 3 side lengths are expressed as $2c$, then the perimeter is $(a + b) + (2c) + (2c) + (2c) = a + b + 6c$.
Finally, if all 4 sides lengths are expressed as $2c$, the perimeter is $(2c) + (2c) + (2c) + (2c) = 8c$.
Of the expressions given, $a + b + 7c$ remains, and since $a + b + 7c = 2c + 7c = 9c$ is not equal to the perimeter, then the correct answer is (B).

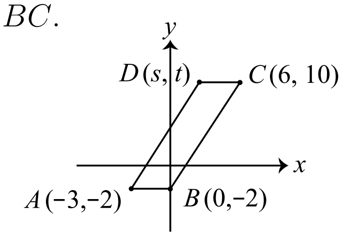
Answer: (B)

21. $H$ is a perfect square and is 1 more than $D$.
The perfect squares from 1 to 8 inclusive are 1 and 4. Since $H$ is one more than $D$, $H$ cannot be equal to 1 (since $D \neq 0$), and so $H = 4$ and $D = 3$.
$B$ is the largest prime number in the set, and so $B = 7$.
$C$ is a multiple of both $G$ and $D$. Since $D = 3$, then $C = 6$ and $G$ is equal to either 1 or 2 (since 6 is a multiple of each of these).
The value of $B + G$ is even, and since $B = 7$, then $G = 1$.
The letters which have not been assigned values are $A$, $E$ and $F$, and the integers which have not been assigned are 2, 5 and 8.
Since 5 and 8 are in the same row and $A$ and $E$ are the two remaining unassigned letters that are in the same row, then $A$ and $E$ are 5 and 8 in some order, which leaves $F = 2$.

Answer: (A)

22. *Solution 1*

The opposite sides of $ABCD$ are parallel, and so $ABCD$ is a parallelogram. The opposite sides of a parallelogram are equal in length, and so $AB = CD$ and $AD = BC$.




Since the value of $r + s + t$ is equal to a constant, we may choose any location for $B(0, r)$ provided that it satisfies the given condition $r < 0$. We choose $r = -2$, so that the $y$-coordinate of $B(0, r)$ is equal to the $y$-coordinate of $A(-3, -2)$, and so $AB$ is a horizontal line segment as shown.

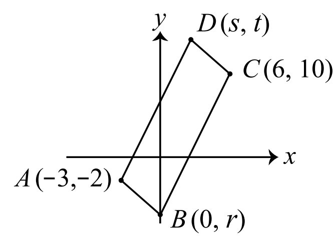
In this case, the length of $AB$ is equal to the positive difference between the $x$-coordinates of $A$ and $B$, which is $0 - (-3) = 3$.

$CD$ is parallel to $AB$, and so $CD$ must also be a horizontal line segment. Thus, points $C(6, 10)$ and $D(s, t)$ have equal $y$-coordinates, and so $t = 10$. Further, $CD$ has the same length as $AB$, and so $6 - s = 3$ or $s = 3$.

Therefore, the value of $r + s + t = -2 + 3 + 10 = 11$.

*Solution 2*

The opposite sides of $ABCD$ are parallel, and so $ABCD$ is a parallelogram. The opposite sides of a parallelogram are equal in length, and so $AB = CD$ and $AD = BC$.




Since $AB$ and $CD$ are parallel and equal in length, then the vertical distance between $A$ and $B$ must equal the vertical distance between $C$ and $D$, and the horizontal distance between $A$ and $B$ must equal the horizontal distance between $C$ and $D$.

The vertical distance between two points is equal to the non-negative difference between their $y$-coordinates, and so $-2 - r = t - 10$ (assuming $r \leq -2$ and $t \geq 10$ as in the diagram).

Simplifying, we get $-2 + 10 = t + r$ and so $t + r = 8$.

The horizontal distance between two points is equal to the non-negative difference between their $x$-coordinates, and so $0 - (-3) = 6 - s$ (assuming $s < 6$ as in the diagram).

Simplifying, we get $0 + 3 = 6 - s$ or $s = 6 - 3$ and so $s = 3$.

Therefore, the value of $r + s + t = (r + t) + s = 8 + 3 = 11$. From Solution 1, we note that $r = -2$, $s = 3$, $t = 10$ are values satisfying the given conditions and for which $r + s + t = 11$.

ANSWER: (B)

23. *Solution 1*

Let the last three digits of $n$ be $abc$. That is, $n$ has units digit $c$, tens digit $b$ and hundreds digit $a$.

(By the end of this solution, we will have demonstrated why considering only the last three digits of $n$ was sufficient.)

The units digit of the product $2013 \times n$ is equal to the units digit of $3 \times c$.

$$\begin{array}{rr} & 2013 \\ \times & abc \\ \hline & \ldots 2025 \end{array}$$

Since the units digit of the product $2013 \times n$ is 5, then the units digit of $3 \times c$ is 5, and so $c = 5$.

(You should confirm for yourself that this is the only possible value of $c$.)

$$\begin{array}{rr} & 2\ 0\ 1\ 3 \\ \times & 5 \\ \hline & 10\ \ 0\ 6\ 5 \end{array}$$

Continuing the long multiplication, the tens digit of $n$ is $b$, and so the tens digit of $2013 \times n$ is equal to the units digit of $6 + 3b$, as shown.

Since the units digit of $6 + 3b$ is 2, then the units digit of $3b$ is 6, and so $b = 2$.

(You should confirm for yourself that this is the only possible value of $b$.)

$$\begin{array}{rr} & 2\ \ 0\ 1\ 3 \\ \times & b\ 5 \\ \hline & 10\ \ 0\ 6\ 5 \\ & \ldots 3b\ \ \\ \hline & \ldots\ 2\ 5 \end{array}$$

The multiplication completed to this point is shown to the right.
We have determined that the last two digits of the product $2013 \times n$ are 25 exactly when the last two digits of $n$ are 25.

$$\begin{array}{r} 2\,0\,1\,3 \\ \times \quad 2\,5 \\ \hline 1\,0\,0\,6\,5 \\ 4\,0\,2\,6\phantom{0} \\ \hline \ldots\,2\,5 \end{array}$$

Continuing the long multiplication, the hundreds digit of $n$ is $a$, and so the hundreds digit of $2013 \times n$ is equal to the units digit of $1+0+2+3a$ (the 1 is the "carry" from the tens column).
Since the units digit of $3+3a$ is 0, then the units digit of $3a$ is 7, and so $a = 9$.
(You should confirm that this is the only possible value of $a$.)

$$\begin{array}{r} 2\,0\,1\,3 \\ \times \quad a\,2\,5 \\ \hline 1\,0\,0\,6\,5 \\ 4\,0\,2\,6\phantom{0} \\ \ldots\,3a\phantom{00} \\ \hline \ldots\;0\,2\,5 \end{array}$$

The last three digits of $2013 \times n$ are 025 exactly when the last three digits of $n$ are 925 (that is, $a = 9$, $b = 2$, $c = 5$ are the only possibilites for $a$, $b$, $c$).
The multiplication completed to this point is shown to the right.
This shows that when $n = 925$, the last four digits of the product $2013 \times n$ are 2025, as required.
Adding additional digits to $n$ will increase the value of $n$, and since we are asked for the smallest possible value of $n$, we stop here.
Thus, the smallest possible value of $n$ for which $2013 \times n$ has last four digits 2025, is $n = 925$, and so the sum of the digits of $n$ is $9+2+5 = 16$.

$$\begin{array}{r} 2\,0\,1\,3 \\ \times \quad 9\,2\,5 \\ \hline 1\,0\,0\,6\,5 \\ 4\,0\,2\,6\phantom{0} \\ 1\,8\,1\,1\,7\phantom{00} \\ \hline 1\,8\,6\,2\,0\,2\,5 \end{array}$$

*Solution 2*
We begin by showing that every positive integer having last two digits 25 is a multiple of 25.
(It is worth noting that it is *not* true that every multiple of 25 has last two digits 25.)
All positive integers whose last two digits are 25, are 25 more than some non-negative multiple of 100.
That is, all positive integers whose last two digits are 25 can be expressed as $100k+25$ for some integer $k \geq 0$.
Since $100k$ is divisible by 25, and 25 is divisible by 25, then $100k + 25$ is divisible by 25.
Thus, every positive integer whose last two digits are 25 is a multiple of 25, and so $2013 \times n$ is a multiple of 25.
Since $2013 = 3 \times 11 \times 61$ does not have a prime factor of 5, then $2013 \times n$ is a multiple of 25 exactly when $n$ is a multiple of 25.
The last two digits of $2013 \times n$ are equal to the two-digit number formed by the last two digits of the product of 13 and the last two digits of $n$.
What are the last two digits of $n$? Since $n$ is a multiple of 25, then the last two digits of $n$ could be 25, 50, 75, or 00.
(You should confirm that these are the only possibilities.)
Thus, the last two digits of $2013 \times n$ are equal to the last two digits of $13 \times 25$ or $13 \times 50$ or $13 \times 75$ or $13 \times 00$, which are 25, 50, 75, and 00 respectively.
Since we require the last two digits of $2013 \times n$ to be 25, then the last two digits of $n$ are 25.
We have reduced the problem to finding the smallest value of the positive integer $n$ with last two digits 25 so that the last four digits of $2013 \times n$ are 2025.
We substitute $n = 25, 125, 225, 325, 425, \ldots$, and so on, in turn, into the product $2013 \times n$.
Evaluating these products, we determine that $2013 \times 925 = 1\,862\,025$ is the first time that the last four digits of $2013 \times n$ are 2025.

Thus, the smallest possible value of $n$ for which $2013 \times n$ has last four digits 2025 is $n = 925$, and the sum of the digits of $n$ is $9 + 2 + 5 = 16$.

ANSWER: (E)

24. Suppose the integer in the centre circle is $a$.
Then each integer in a circle connected to the centre circle is either $d$ more than $a$, which is $a + d$, or it is $d$ less than $a$, which is $a - d$.
The integers in the two circles connected to the centre could both be $a+d$, as in Figure 1 below, or they could both be $a - d$, as in Figure 2, or one could be $a - d$ and one could be $a + d$, as in Figure 3.
We note that in the last case (Figure 3), swapping locations of the $a - d$ and the $a + d$ does not change the integers in the final two empty circles (since they still depend on $a + d$ and $a - d$), and thus does not change the sum of the five integers.

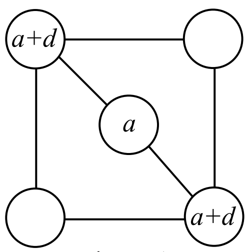


Figure 1

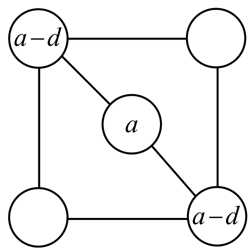


Figure 2

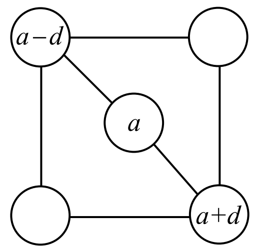


Figure 3

Next, we explain why it is possible to place integers into the remaining two circles (in each of the three cases above) so that the positive difference between each pair of integers in connected circles is $d$.
For the case that began in Figure 1, the integers in the two empty circles are either $d$ more than $a + d$, which is $a + 2d$, or they are $d$ less than $a + d$, which is $a$.
These integers could both be $a + 2d$, as in Figure 1a below, or they could both be $a$, as in Figure 1b, or one could be $a + 2d$ and one could be $a$, as in Figure 1c.
We note that in the last case (Figure 1c), swapping locations of the final two integers, $a + 2d$ and $a$, does not change the sum of the five integers.

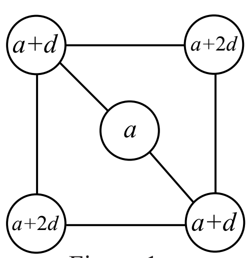


Figure 1a

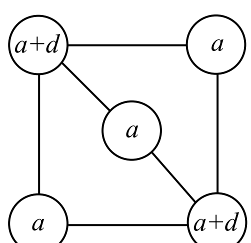


Figure 1b

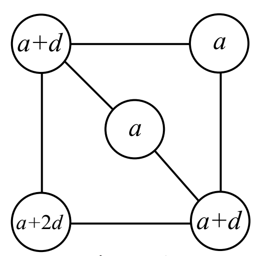


Figure 1c

For the case that began in Figure 2, the integers in the two empty circles are either $d$ more than $a - d$, which is $a$, or they are $d$ less than $a - d$, which is $a - 2d$.
These integers could both be $a$, as in Figure 2a below, or they could both be $a - 2d$, as in Figure 2b, or one could be $a$ and one could be $a - 2d$, as in Figure 2c.
We again note that in the last case (Figure 2c), swapping locations of the final two integers, $a$ and $a - 2d$, does not change the sum of the five integers.

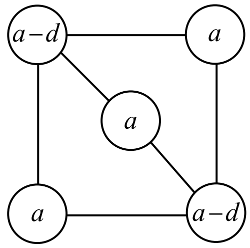



Figure 2a

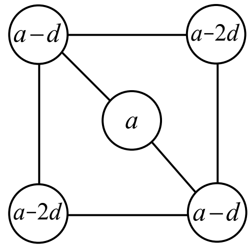



Figure 2b

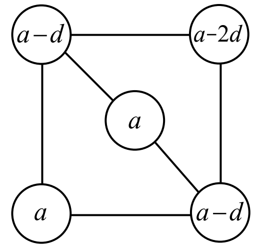



Figure 2c

Finally, for the case that began in Figure 3, each integer in an empty circle must have a positive difference of $d$ with both $a-d$ and $a+d$.
The integer $d$ more than $a-d$ is $a$, and the integer $d$ less than $a-d$ is $a-2d$.
The integer $d$ more than $a+d$ is $a+2d$, and the integer $d$ less than $a+d$ is $a$.
Thus, $a$ is the only integer that has a positive difference of $d$ with both $a-d$ and $a+d$, and so the integers in the two empty circles must each be equal to $a$, as shown in Figure 3a.

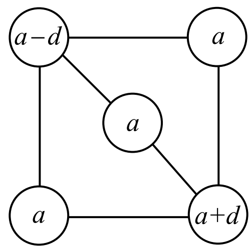
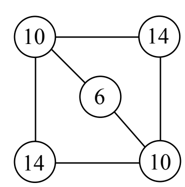



Figure 3a

Suppose that the sum of the five integers in the circles is $S$.
For Case 1a (which corresponds to Figure 1a), adding the five integers in the figure, we get $S = a + (a + d) + (a + 2d) + (a + d) + (a + 2d) = 5a + 6d$.
In the table below, we determine the value of $S$ for each of the 7 cases.

| Case 1a | Case 1b | Case 1c | Case 2a | Case 2b | Case 2c | Case 3a |
|---|---|---|---|---|---|---|
| $S = 5a + 6d$ | $S = 5a + 2d$ | $S = 5a + 4d$ | $S = 5a - 2d$ | $S = 5a - 6d$ | $S = 5a - 4d$ | $S = 5a$ |

We must determine the number of different integers $d$, between 1 and 20 inclusive, for which at least one of the seven expressions for $S$ is equal to 54 and $a$ is an integer.
Consider Case 1a, from which we get $5a + 6d = 54$.
Since both $a$ and $d$ are integers, and $d$ is between 1 and 20 inclusive, we can systematically substitute values of $d$ into this equation, and then solve for $a$ to determine if $a$ is an integer.
For example if $d = 1$, we get $5a + 6 \times 1 = 54$ or $5a = 48$.
However, there is no integer $a$ for which $5a = 48$ and so $d = 1$ is not a possible value of $d$ in Case 1a. Substituting $d = 2$ and $d = 3$ similarly give non-integer values of $a$.
When $d = 4$, we get $5a + 6 \times 4 = 54$ and so $5a = 30$ or $a = 6$.
In this case, the pair of integers $d = 4$ and $a = 6$ satisfy the equation $5a + 6d = 54$.
Substituting $d = 4$ and $a = 6$ into Figure 1a, we get the diagram shown to the right.




We can confirm that the positive difference between each pair of integers in connected circles is 4 (an integer between 1 and 20 inclusive), and the sum of the five integers in the circles is 54, as required. Thus $d = 4$ is a possible value satisfying the given conditions.
We can systematically continue to substitute $d = 5, 6, 7, \ldots, 20$ into $5a + 6d = 54$ and solve the

equation to determine which values of $d$ give integer values of $a$.
The next smallest value of $d$ for which $a$ is an integer is $d = 9$. In this case, we get $5a+6\times 9 = 54$ and so $5a = 0$ or $a = 0$.
We could continue in this systematic way, however since there are 20 possible values of $d$ and 7 cases to check, this would take a while to complete. Instead, we might recognize that $d = 4$, $a = 6$ and $d = 9$, $a = 0$ are both solutions to $5a + 6d = 54$.
Notice that from the first solution to the second, the value of $d$ increases by 5, and the value of $a$ decreases by 6.
Can you see why increasing $d$ by 5 and decreasing $a$ by 6 gives the next possible pair of integers for which $5a + 6d = 54$? (Hint: Take a close look at the left side of the equation.)
If we increase $d$ by 5 again, and decrease $a$ by 6, we get $d = 9 + 5 = 14$ and $a = 0 - 6 = -6$, and since $5a + 6d = 5 \times (-6) + 6 \times 14 = -30 + 84 = 54$, then $d = 14$ and $a = -6$ is a solution to the equation (and in fact, this is the next smallest value of $d$ that works).
The final integer value of $d$ between 1 and 20 inclusive for which $5a+6d = 54$ is $d = 14+5 = 19$, and in this case $a = -6 - 6 = -12$ or $(d, a) = (19, -12)$
Therefore, Case 1a gives $d = 4, 9, 14,$ and 19, or 4 values of $d$ which satisfy the given conditions.

We continue in this way for each of the first four cases, and summarize all possible integer solutions for those cases in the table below.

| | | | |
|---|---|---|---|
| Case 1a | $5a + 6d = 54$ | $(d, a) = (4, 6), (9, 0), (14, -6), (19, -12)$ | $d = 4, 9, 14, 19$ |
| Case 1b | $5a + 2d = 54$ | $(d, a) = (2, 10), (7, 8), (12, 6), (17, 4)$ | $d = 2, 7, 12, 17$ |
| Case 1c | $5a + 4d = 54$ | $(d, a) = (1, 10), (6, 6), (11, 2), (16, -2)$ | $d = 1, 6, 11, 16$ |
| Case 2a | $5a - 2d = 54$ | $(d, a) = (3, 12), (8, 14), (13, 16), (18, 18)$ | $d = 3, 8, 13, 18$ |

Notice that after the first four cases shown above, all possible values of $d$ from 1 to 20 inclusive satisfy the given conditions with the exception of $d = 5, 10, 15,$ and 20.
In Case 3a we get, $5a = 54$ and so $a$ is not an integer. Next, consider Case 2b, $5a - 6d = 54$.
Each value of $d$ left to check ($d = 5, 10, 15, 20$) is a multiple of 5, and so $6d$ is a multiple of 5 for each of these possible values of $d$.
Since $5a$ is also a multiple of 5 for all possible integers $a$, then $5a - 6d$ is the difference between two multiples of 5, and thus is a multiple of 5.
However, the right side of the equation $5a - 6d = 54$ is not a multiple of 5 and so $d$ cannot be equal to a multiple of 5.
In the final case, $5a - 4d = 54$, it is similarly not possible for $d$ to be equal to a multiple of 5.
Thus $d$ can be equal to each of the first 20 positive integers with the exception of $5, 10, 15,$ and 20, and so there are $20 - 4 = 16$ different possible values of $d$.

It is worth noting that there are many different ways to find the integer solutions to each of the 7 equations (cases) above. For example, the value of each of the terms $6d, 2d, 4d, -2d, -6d, -4d, 0d$ is even for all integers $d$, and the right side of each equation, 54, is also even. This means that in each equation, the value of $5a$ must be even, and so $a$ is even. Further, when $a$ is even, the units digit of $5a$ is 0. Since the units digit of 54 is 4, what do we now know about the units digit of each term containing a $d$, and in each case, what does that tell us about the possible values of $d$?

ANSWER: (E)

25. We begin by recognizing that in the given list, each of the digits 1 through 7 occurs at least once as a units digit, and at least once as a tens digit.
For example, the digit 1 occurs twice as a units digit (11 and 31), and three times as a tens digit (11, 12 and 14).
Counting the number of times each of the digits 1 through 7 occurs as a units digit and as a tens digit, we get:

| Digit | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Number of times occuring as a units digit | 2 | 1 | 1 | 4 | 1 | 2 | 1 |
| Number of times occuring as a tens digit | 3 | 1 | 1 | 3 | 1 | 2 | 1 |

The units digit of each number in the list matches the tens digit of the number that follows it. This tells us that if we ignore the tens digit of the first number in the list and the units digit of the last number in the list, then the number of times that each digit occurs as a units digit must be equal to the number of times that it occurs as a tens digit.
Looking back to the table above, we see that this is true for all digits except 1 and 4.
Since the digit 1 occurs twice as a units digit and three times as a tens digit, then the tens digit of the first number in the list must be equal to 1.
Similarly, the digit 4 occurs four times as a units digit and three times as a tens digit, and so the units digit of the last number in the list must be equal to 4.
Ignoring the number 14 for a moment, we separate the 11 remaining numbers into two distinct lists, which we call $A$ and $B$.

$$A: 11, 12, 23, 31 \qquad B: 44, 45, 46, 56, 64, 67, 74$$

Each digit in $A$ is less than or equal to 3, and each digit in $B$ is greater than or equal to 4.
Since 14 is the only number given that does not appear in $A$ or $B$, and 14 has a digit that appears in $A$ and a digit that appears in $B$, then 14 is the only number that can 'connect' the numbers in $A$ to those in $B$.
Further, this tells us that the numbers in $A$ must be arranged and then placed before an arrangement of the numbers in $B$, with 14 appearing between the two arrangements.
Also, the arrangement of the numbers in $A$ must begin and end with a 1, and the arrangement of the numbers in $B$ must begin and end with a 4 (since 14 occurs between the two lists).

Next, we count the number of different ways to arrange the numbers in $A$, starting and ending with 1.
We begin by recognizing that each of the digits 2 and 3 occurs exactly once as a units digit and once as a tens digit, and so 12, 23, 31 must appear together in this order (the two 2s must occur together and the two 3s must occur together).
The list must begin and end with a 1, and so there are 2 possible locations for the 11 and thus 2 possible arrangements of the numbers in $A$: 11, 12, 23, 31, and 12, 23, 31, 11.

Next, we count the number of different ways to arrange the numbers in $B$, starting and ending with 4.
We begin by recognizing that each of the digits 5 and 7 occurs exactly once as a units digit and once as a tens digit, and so 45, 56 must appear together in this order (the two 5s must occur together), and 67, 74 must appear together in this order (the two 7s must occur together).
The arrangement ends with a 4, and thus cannot end with 45, 56, and so at least one more number must immediately follow 45, 56.
There are two such possibilities: 45, 56, 64, and 45, 56, 67, 74 (recall that 67, 74 must remain together), which leads to exactly two distinct cases to consider.

Case 1: 45, 56, 64 occur together in this order

In this case, the remaining numbers are 44, 46, 67, and 74.
Since 44 has two equal digits, its location in the arrangement of the $B$ list cannot change the first digit in the list (which must be 4), and cannot change the last digit in the list (which must also be 4), and thus we ignore 44 for the moment.
The remaining numbers, 46, 67, 74 must occur together in this order. Can you see why?
Since the blocks 45, 56, 64 and 46, 67, 74 must each occur together in their respective orders, this gives two possible arrangements of the $B$ list (ignoring the 44).
These are: 45, 56, 64, 46, 67, 74, and 46, 67, 74, 45, 56, 64.
Next, we determine the number of different ways to place 44 into each of these arrangements.
In the 45, 56, 64, 46, 67, 74 arrangement, the 44 may appear at the start, at the end, or between the 64 and 46, which gives 3 different arrangements of the $B$ list. (These are: 44, 45, 56, 64, 46, 67, 74, and 45, 56, 64, 46, 67, 74, 44, and 45, 56, 64, 44, 46, 67, 74.)
In the 46, 67, 74, 45, 56, 64 arrangement, the 44 may appear at the start, at the end, or between the 74 and 45, which gives 3 more arrangements of the $B$ list, or 6 in total for Case 1.

Case 2: 45, 56, 67, 74 occur together in this order

In this case, the remaining numbers are 44, 46, and 64.
We again begin by ignoring 44 for the moment.
The remaining numbers, 46, 64 must occur together in this order.
Since the blocks 45, 56, 67, 74 and 46, 64 must each occur together in their respective orders, this gives two possible arrangements of the $B$ list (ignoring the 44).
These are: 45, 56, 67, 74, 46, 64 and 46, 64, 45, 56, 67, 74.
In the 45, 56, 67, 74, 46, 64 arrangement, the 44 may appear at the start, at the end, or between the 74 and 46, which gives 3 more different arrangements of the $B$ list.
In the 46, 64, 45, 56, 67, 74 arrangement, the 44 may appear at the start, at the end, or between the 64 and 45, which gives 3 more arrangements of the $B$ list, or 6 in total for Case 2.

Thus, there are a total of $6 + 6 = 12$ different ways to arrange the $B$ list.

There are 2 different ways to arrange the numbers in list $A$, 12 different ways to arrange the numbers in list $B$, and exactly 1 way to place the number 14 between arrangements of each of the two lists. Thus, the total number of arrangements of the given list is $2 \times 12 \times 1 = 24$.

ANSWER: (B)