The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2025 Galois Contest

Thursday, April 3, 2025
(in North America and South America)

Friday, April 4, 2025
(outside of North America and South America)

Solutions

1. (a) Reading from the graph, there is 1 student who is 14, 0 students who are 15, 3 students who are 16, and 2 students who are 17.
The total number of students in the Art Club is $1 + 0 + 3 + 2 = 6$.

(b) Using the work from part (a), the mean (average) age of the students in the Art Club is $\dfrac{(1 \times 14) + (3 \times 16) + (2 \times 17)}{6} = \dfrac{96}{6} = 16$.

(c) From part (b), the sum of the ages of the 6 students currently in the Art Club is 96.
If n 15-year old students join the club, the number of students in the club will be $6 + n$, and the sum of their ages will be $96 + 15n$.
When n 15-year old students join the club, the mean age of all students in the club is 15.5.
Solving for n, we get

$$\begin{aligned} \frac{96 + 15n}{6 + n} &= 15.5 \\ 96 + 15n &= 15.5(6 + n) \\ 96 + 15n &= 93 + 15.5n \\ 3 &= 0.5n \end{aligned}$$

and so $n = \dfrac{3}{0.5} = 6$. Therefore, the number of 15-year old students that must join the Art Club so that the mean age of the students in the club is 15.5 is 6.

2. For each part, the completed magic square is shown following part (d).

(a) The magic constant is 18, and so the missing number in the first row is $18 - 7 - 2 = 9$. Looking at the diagonal from the top-right corner to the bottom-left corner, we get $9 + n + 3 = 18$, and so $n = 6$.

7	2	
	n	
3		

(b) Reading from the first column, the magic constant is $8 + 9 + 4 = 21$. Thus, the missing number in the second row is $21 - 9 - 5 = 7$. Looking at the diagonal from the top-right corner to the bottom-left corner, the missing number in the top-right corner is $21 - 7 - 4 = 10$.
From the first row, we get $8 + p + 10 = 21$, and so $p = 3$.

8	p	
9		5
4		

(c) *Solution 1*
The sum of the numbers in the first column is equal to the sum of the numbers in the third row. Since these two sums both share the missing number in the bottom-left corner, then the sum of the remaining two numbers in the first column must equal the sum of the remaining two numbers in the third row. That is, $13 + 7 = (r + 1) + (r + 3)$ and so $20 = 2r + 4$ or $16 = 2r$, which gives $r = 8$.

13		r
7		17
	$r+1$	$r+3$

Solution 2
The sum of the numbers in the third column is $r + 17 + (r + 3) = 2r + 20$, and so the sum of the numbers in the third row is also $2r + 20$. Thus, the missing number in the third row is $(2r + 20) - (r + 1) - (r + 3) = 16$.
From the first column, the magic constant is $13 + 7 + 16 = 36$, and so $2r + 20 = 36$ or $2r = 16$, which gives $r = 8$.

(d) The sum of the numbers in the third row is $(u+2)+(u-5)+u = 3u-3$, and so the sum of the numbers in the second column is also $3u-3$. Thus, the missing number in the second column is $(3u-3)-(u+3)-(u-5) = u-1$, as shown.

	$u+3$	
	$u-1$	12
$u+2$	$u-5$	u

The sum of the numbers in the diagonal from the top-right corner to the bottom-left corner is equal to the sum of the numbers in the third column. Since these two sums both share the missing number in the top-right corner, then the sum of the remaining two numbers in the diagonal must equal the sum of the remaining two numbers in the third column.
That is, $(u+2)+(u-1) = u+12$ or $2u+1 = u+12$, and so $u = 11$.

(a)

7	2	9
8	6	4
3	10	5

(b)

8	3	10
9	7	5
4	11	6

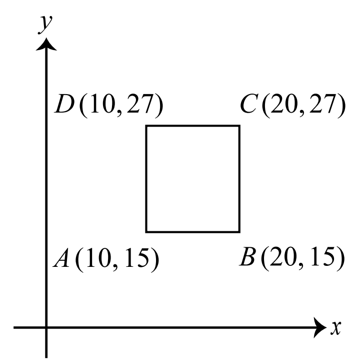
(c)

13	15	8
7	12	17
16	9	11

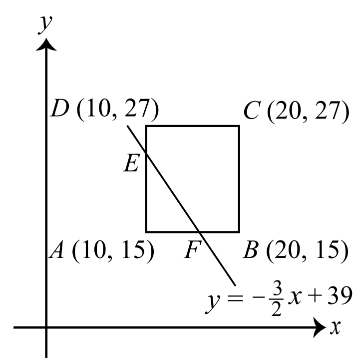
(d)

9	14	7
8	10	12
13	6	11

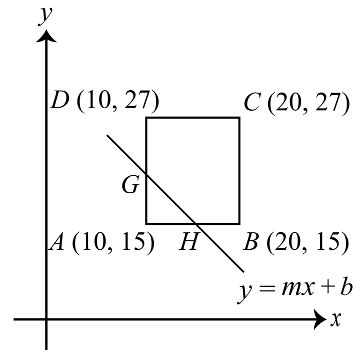
3. (a) In rectangle $ABCD$, AD is parallel to the y-axis and so opposite side BC is also parallel to the y-axis.
This means that sides AD and BC are vertical, so A and D each have the same x-coordinate, 10, and B and C also have the same x-coordinate, 20.
Similarly, AB is parallel to the x-axis and so CD is also parallel to the x-axis.
Thus, sides AB and CD are horizontal, so A and B have the same y-coordinate, 15, and C and D also have the same y-coordinate, 27.
Therefore, the coordinates of B are $(20, 15)$, and the coordinates of D are $(10, 27)$, as shown.
Since $AB = 20-10 = 10$ and $BC = 27-15 = 12$, the area of rectangle $ABCD$ is $10 \times 12 = 120$.



(b) Point E lies on AD as shown and thus has x-coordinate 10.
Point E lies on the line with equation $y = -\frac{3}{2}x + 39$, and so when $x = 10$, the y-coordinate is $y = -\frac{3}{2}(10) + 39 = 24$.
Point F lies on AB and thus has y-coordinate 15.
Point F lies on the line with equation $y = -\frac{3}{2}x + 39$, and so when $y = 15$, we get $15 = -\frac{3}{2}x + 39$ or $-24 = -\frac{3}{2}x$, and so $x = 24 \times \frac{2}{3} = 16$.
Point E has coordinates $(10, 24)$, and so $EA = 24 - 15 = 9$.
Point F has coordinates $(16, 15)$, and so $AF = 16 - 10 = 6$.
The area of $\triangle EAF$ is $\frac{1}{2}(EA)(AF) = \frac{1}{2}(9)(6) = 27$.
The area of pentagon $BCDEF$ is the area of $ABCD$ minus the area of $\triangle EAF$, which is $120 - 27 = 93$.



(c) *Solution 1*

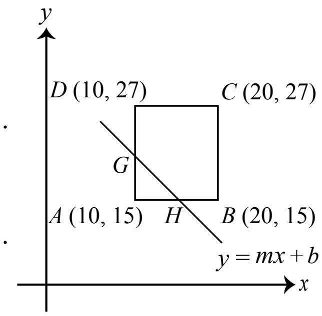


Point G lies on AD as shown and thus has x-coordinate 10.
Point G lies on the line with equation $y = mx + b$, and so when $x = 10$, the y-coordinate is $y = 10m + b$.
Point H lies on AB as shown and thus has y-coordinate 15.
Point H lies on the line with equation $y = mx + b$, and so when $y = 15$, we get $15 = mx + b$ or $15 - b = mx$.
Therefore, $x = \dfrac{15-b}{m}$.
Point G has coordinates $(10, 10m + b)$, and so $GA = 10m + b - 15$.
Point H has coordinates $\left(\dfrac{15-b}{m}, 15\right)$, and so $AH = \dfrac{15-b}{m} - 10$.
The area of $\triangle GAH$ is $\frac{1}{2}(GA)(AH) = \frac{1}{2}(10m + b - 15)\left(\dfrac{15-b}{m} - 10\right)$.
Setting the area of $\triangle GAH$ equal to $-\dfrac{8}{m}$ and simplifying, we get

$$\begin{aligned}
\frac{1}{2}(10m+b-15)\left(\frac{15-b}{m}-10\right) &= -\frac{8}{m} \\
2m \times \frac{1}{2}(10m+b-15)\left(\frac{15-b}{m}-10\right) &= -\frac{8}{m} \times 2m \qquad \text{(since } m \neq 0\text{)} \\
(10m+b-15)\left(m \times \left(\frac{15-b}{m}-10\right)\right) &= -16 \\
(10m+b-15)(15-b-10m) &= -16 \\
(10m+b-15)^2 &= 16 \\
10m+b-15 &= \pm 4
\end{aligned}$$

and so $10m + b = 11$ or $10m + b = 19$.
Since $10m + b$ is the y-coordinate of G, and G lies between $A(10, 15)$ and $D(10, 27)$, then $15 < 10m + b < 27$ and so $10m + b \neq 11$.
Both m and b are integers with $m < 0$ and $b < 50$.
Suppose that $m = -1$, the largest possible value of m.
In this case, $10(-1) + b = 19$ and so $b = 29$. When $(m, b) = (-1, 29)$, we confirm that the x-coordinate of H is $x = \dfrac{15-b}{m} = \dfrac{15-29}{-1} = 14$, and so H lies between A and B.
If $m = -2$, then $10(-2) + b = 19$ and so $b = 39$. When $(m, b) = (-2, 39)$, the x-coordinate of H is $x = \dfrac{15-39}{-2} = 12$, and so H lies between A and B.
If $m = -3$, then $10(-3) + b = 19$ and so $b = 49$. When $(m, b) = (-3, 49)$, the x-coordinate of H is $x = \dfrac{15-49}{-3} = \dfrac{34}{3}$, and so H lies between A and B.
If $m \leq -4$, then $b = 19 - 10m \geq 19 - 10(-4) = 59$ which is not possible since $b < 50$.
The ordered pairs of integers (m, b) with $b < 50$, and for which the area of $\triangle GAH$ is equal to $-\dfrac{8}{m}$, are $(-1, 29), (-2, 39)$ and $(-3, 49)$.

Solution 2



Point G lies on AD as shown and thus has x-coordinate 10.
Suppose the y-coordinate of G is g.
Since G lies between A and D, then $15 < g < 27$ and $GA = g - 15$.
Point H lies on AB as shown and thus has y-coordinate 15.
Suppose the x-coordinate of H is h.
Since H lies between A and B, then $10 < h < 20$ and $AH = h - 10$.
The area of $\triangle GAH$ is $\frac{1}{2}(GA)(AH) = \frac{1}{2}(g-15)(h-10)$.
The area of $\triangle GAH$ is also equal to $-\frac{8}{m}$, and so $\frac{1}{2}(g-15)(h-10) = -\frac{8}{m}$ or $(g-15)(h-10) = -\frac{16}{m}$.
The line through $G(10, g)$ and $H(h, 15)$ has slope $\frac{g-15}{10-h}$.
The line through G and H has equation $y = mx + b$ and thus slope m.
Equating slopes, we get $\frac{g-15}{10-h} = m$.
Using the equations $\frac{g-15}{10-h} = m$ and $(g-15)(h-10) = -\frac{16}{m}$, along with the property that if $p = q$ and $r = s$, then $p \times r = q \times s$, we get the following:

$$\begin{aligned} \frac{g-15}{10-h} \times (g-15)(h-10) &= m \times -\frac{16}{m} \\ -\frac{g-15}{h-10} \times (g-15)(h-10) &= m \times -\frac{16}{m} \\ (g-15)^2 &= 16 \quad \text{(since both } m \text{ and } h-10 \text{ are not 0)} \\ g-15 &= \pm 4 \end{aligned}$$

and so $g = 15 + 4 = 19$ or $g = 15 - 4 = 11$.
Since $15 < g < 27$, then $g = 19$.
The line with equation $y = mx + b$ passes through $G(10, 19)$ and so $19 = 10m + b$ or $b = 19 - 10m$.
Both m and b are integers with $m < 0$ and $b < 50$.
Suppose that $m = -1$, the largest possible value of m.
In this case, $b = 19 - 10(-1) = 29$. When $(m, b) = (-1, 29)$, we use the equation of the line $y = mx + b$ to confirm that the x-coordinate of $H(h, 15)$ is $h = \frac{15-b}{m} = \frac{15-29}{-1} = 14$, and so $H(14, 15)$ lies between $A(10, 15)$ and $B(20, 15)$, as required.
If $m = -2$, then $b = 19 - 10(-2) = 39$. When $(m, b) = (-2, 39)$, we can similarly show that $h = 12$ and so H lies between A and B.
If $m = -3$, then $b = 19 - 10(-3) = 49$. When $(m, b) = (-3, 49)$, we get $h = \frac{34}{3}$ and so H lies between A and B.
If $m \leq -4$, then $b = 19 - 10m \geq 19 - 10(-4) = 59$ which is not possible since $b < 50$.
The ordered pairs of integers (m, b) with $b < 50$, and for which the area of $\triangle GAH$ is equal to $-\frac{8}{m}$, are $(-1, 29), (-2, 39)$ and $(-3, 49)$.

4. (a) The prime factorization of 400 is $2^4 5^2$, and so 400 has $(4+1)(2+1) = 15$ positive divisors. Since $20^2 = 400$, one of these 15 positive divisors is 20, from which we get the ordered pair $(a, b) = (20, 20)$.

The remaining 14 positive divisors give $\frac{14}{2} = 7$ factor pairs of positive integers (a, b) with $a < b$. One of these 7 factor pairs has $a = 1$, which we must omit since $a > 1$.
Thus, there are 6 ordered pairs (a, b) with $a < b$ and 1 with $a = b$, for a total of 7 ordered pairs of positive integers (a, b) for which $ab = 400$ and $1 < a \leq b$.
(These ordered pairs are $(2, 200)$, $(4,100), (5, 80)$, $(8, 50)$, $(10, 40)$, $(16, 25)$, and $(20, 20)$.)

(b) The prime factorization of 270 000 is $2^4 3^3 5^4$.
We count the number of ways to distribute each of the prime factors among p, q and r.

The three prime factors equal to 3 could all be distributed to exactly one of p, q or r. This can be done in 3 different ways.
Two 3s can be distributed to one factor, and one 3 to another. There are 3 choices for the factor having two 3s, and 2 choices for the factor having one 3, and thus $3 \times 2 = 6$ ways to distribute two 3s to one factor and one 3 to another.
Finally, one 3 can be distributed to each of the factors, and this can be done in 1 way. Thus, there are $3 + 6 + 1 = 10$ ways to distribute the 3s among p, q and r.

The four prime factors equal to 2 could all be distributed to exactly one of p, q or r. This can be done in 3 different ways.
Three 2s can be distributed to one factor, and one 2 to another. There are 3 choices for the factor having three 2s, and 2 choices for the factor having one 2, and thus $3 \times 2 = 6$ ways to distribute three 2s to one factor and one 2 to another.
Two 2s can be distributed to one factor, and two 2s to another factor. There are 3 choices for the factor having two 2s, and 2 choices for the second factor having two 2s. However, since the number of 2s being distributed to each factor is equal, we have double counted the possibilities.
Thus, there are $\frac{3 \times 2}{2} = 3$ ways to distribute two 2s to one factor and two 2s to another.
(This is equivalent to counting the number of ways of choosing the factor receiving no 2s.)
Finally, two 2s can be distributed to one of the factors in 3 ways, and one 2 can be distributed to each of the other two factors in 1 way, for 3 possibilities in this final distribution. Thus, there are $3 + 6 + 3 + 3 = 15$ ways to distribute the 2s among p, q and r.

Similarly, there are 15 ways to distribute the four 5s among p, q and r, and so there are $10 \times 15 \times 15 = 2250$ ways to distribute the prime factors, and thus 2250 ordered triples of positive integers (p, q, r) for which $pqr = 270\,000$.

(c) As shown in part (b), there are 2250 ordered triples of positive integers (x, y, z) for which $xyz = 270\,000 = 2^4 3^3 5^4$.
For distinct values of x, y and z, these 2250 triples include the 6 possible arrangements of x, y and z.
For example, each of the 6 arrangements of $(5^4, 2^4, 3^3)$ is included among the 2250 triples. Of these 6, it is only $(2^4, 3^3, 5^4)$ that is counted in part (c) since we require $x \leq y \leq z$.
Thus, we must determine the number of ordered triples from part (b) which do not satisfy $x \leq y \leq z$ and subtract this from 2250 (we refer to this as Step 1).
Further, we require ordered triples for which $1 < x$. Thus, we also must determine the number of ordered triples for which $x = 1$ and subtract this from the number of triples that remain following Step 1. We refer to this as Step 2.

Step 1

Since 270 000 is not a perfect cube, it is not possible that $x = y = z$.
Thus, each of the 2250 ordered triples belongs to exactly one of the following two cases: exactly two of x, y, z are equal, or all three are distinct.
Suppose that exactly two of the factors are equal, say $x = y$.
In this case $xyz = x^2z = 2^4 3^3 5^4$, and so x^2 is a perfect square divisor of $2^4 3^3 5^4$.
Each perfect square divisor of $2^4 3^3 5^4$ is a number of the form $2^u 3^v 5^w$ where $u = 0, 2$ or 4, $v = 0$ or 2, and $w = 0, 2$ or 4.
There are 3 choices for u, 2 choices for v, and 3 choices for w, and so there are $3 \times 2 \times 3 = 18$ perfect square divisors of $2^4 3^3 5^4$.
When exactly two of the factors are equal, there are 3 ways to arrange the three factors, and so there are $3 \times 18 = 54$ ordered triples for which exactly two of x, y, z are equal.
Thus, the number of ordered triples for which x, y and z are distinct is $2250 - 54 = 2196$.
For distinct values of x, y and z, the 2196 triples include each of the 6 possible arrangements of x, y and z.
Thus, the number of ordered triples (x, y, z) with $1 \le x < y < z$ is $\dfrac{2196}{6} = 366$.
Since there are 18 ordered triples (x, y, z) for which x, y, z are not distinct (exactly two are equal), then there are $366 + 18 = 384$ ordered triples (x, y, z) with $1 \le x \le y \le z$.
This completes Step 1.

Step 2

We require each of x, y, z to be greater than 1, and so in this final step we determine the number of ordered triples for which $x = 1$, and subtract this from 384.
When $x = 1$, $xyz = 2^4 3^3 5^4$ becomes $yz = 2^4 3^3 5^4$ which means that (y, z) is a factor pair of $2^4 3^3 5^4$.
Since $2^4 3^3 5^4$ has $5 \times 4 \times 5 = 100$ positive divisors, then it has $\dfrac{100}{2} = 50$ factor pairs of positive integers (y, z) with $y \le z$, and so there are 50 ordered triples for which $x = 1$.
Thus, the number of ordered triples of positive integers (x, y, z) for which $xyz = 270\,000$ and $1 < x \le y \le z$ is $384 - 50 = 334$.