The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2025 Fryer Contest

Thursday, April 3, 2025
(in North America and South America)

Friday, April 4, 2025
(outside of North America and South America)

Solutions

1. (a) During the first week, Azizi sold a total of $15+23+18+15+7=78$ chocolate bars.

   (b) During the second week, Azizi sold a total of $7+15+x+23+x=45+2x$ chocolate bars, and so $45+2x=73$. Solving $45+2x=73$, we get $2x=28$, and so $x=14$.

   (c) During the third week, Azizi sold y chocolate bars on Monday, $y+6$ on Tuesday, $y+12$ on Wednesday, $y+18$ on Thursday, and $y+24$ on Friday.
   In total, he sold $y+(y+6)+(y+12)+(y+18)+(y+24)=5y+60$ chocolate bars, and so $5y+60=100$. Solving $5y+60=100$, we get $5y=40$, and so $y=8$.
   (Can you explain why the total number sold, $5y+60$, is equal to 5 times the number that Azizi sold on Wednesday?)

2. (a) We organize the given results in a table similar to the example shown. A winning player is awarded $W=3$ points for each game won.

Game Results	Points Awarded		
	Ava	Beau	Cato
Ava and Beau tie	1	1	——
Beau loses to Cato	——	0	3
Ava and Cato tie	1	——	1

   When the tournament has finished, Ava has been awarded 2 points, Beau has been awarded 1 point, and Cato has been awarded 4 points, and so $S=2+1+4=7$.

   Alternately, 2 games end in a tie, and in each game ending in a tie, 2 points are awarded (1 point to each player). Thus, the 2 tie games contribute $2\times 2=4$ points to S. The remaining game ended with a winner, and so 3 points are awarded (3 to the winning player, 0 to the losing player). Thus, this game contributes 3 points to S, and so $S=4+3=7$.

   (b) Each game that ends in a tie contributes 2 points to S (1 point to each player).
   If all 3 games end in a tie, then $S=3\times 2=6$, as required.

   Next, we confirm that all 3 games ending in a tie is the only possibility.
   Each game that ends with a winner contributes 4 points to S (4 to the winning player, 0 to the losing player).
   If all 3 games end with a player winning, then $S=3\times 4=12$.
   If 2 games end with a player winning, and 1 game ends in a tie, then $S=2\times 4+2=10$.
   If 1 game ends with a player winning, and 2 games end in a tie, then $S=4+2\times 2=8$.
   Thus, if $W=4$ and $S=6$, the only possibility is that all 3 games end in a tie.

   (c) The tournament ends with exactly one of the three games ending in a tie, and so exactly two of the games end with a player winning.
   The game ending in a tie contributes 2 to S (1 point to each player).
   A game that ends with a player winning contributes W to S, and so two games that end with a player winning contribute $2W$ to S.
   In this case, $S=2W+2$ or $2W+2=24$ which gives $2W=22$, and so $W=11$.

   (d) As was demonstrated in (b), there are four outcomes that must be considered when determining the possible values of S. The tournament can end with exactly 0, 1, 2, or 3 ties.
   If the tournament ends with 0 ties, then each of the 3 games has a winning player, and thus $S=3W$. In this case, $3W=21$ and so $W=7$.
   If the tournament ends with 1 tie, then 2 games have a winning player, and thus $S=2W+2$. In this case, $2W+2=21$ or $2W=19$ which is not possible since W is an integer.

If the tournament ends with 2 ties, then 1 game has a winning player, and thus $S = W + 2 \times 2$. In this case, $W + 4 = 21$ or $W = 17$.
Finally, if the tournament ends with 3 ties, then 0 games have a winning player, and thus $S = 3 \times 2 = 6$, and so S cannot equal 21.
If the tournament finishes with $S = 21$, the possible integer values of W are 7 and 17.

3. (a) A positive integer $n > 1$ is a perfect square exactly when the exponent on each prime factor in the prime factorization of n is even.
We note that $2^3 \times 3^2$ has an odd exponent on the prime factor 2, and an even exponent on the prime factor 3.
Thus, $2^3 \times 3^2 \times j$ is a perfect square exactly when j is equal to the product of an odd number of factors of 2 and an even number of each additional prime factor greater than 2 (including possibly having no additional prime factors).
Since $j \leq 20$, then the prime factorization of j must contain either one factor of 2 or three factors of 2, and j cannot contain five or more factors of 2 since $2^5 > 20$.
When $j = 2$, the given product, $2^3 \times 3^2 \times 2 = 2^4 \times 3^2 = (2^2 \times 3) \times (2^2 \times 3)$, is a perfect square.
When $j = 2^3 = 8$, the given product, $2^3 \times 3^2 \times 2^3 = 2^6 \times 3^2 = (2^3 \times 3) \times (2^3 \times 3)$, is also a perfect square.
Is it possible for the prime factorization of j to contain one 2 and an even number of another prime factor greater than 2?
The smallest prime number greater than 2 is 3, and when $j = 2 \times 3^2 = 18 < 20$, the given product $2^3 \times 3^2 \times 2 \times 3^2 = 2^4 \times 3^4 = (2^2 \times 3^2) \times (2^2 \times 3^2)$, is a perfect square.
The next smallest possible value of j for which its prime factorization contains exactly one 2 is 2×5^2, which is greater than 20. (Note that $j = 2 \times 3^4$ is also greater than 20.)
The next smallest possible value of j for which its prime factorization contains exactly three factors of 2 is $2^3 \times 3^2$, which is also greater than 20.
Therefore, the positive integers j which satisfy the given conditions are 2, 8 and 18.

(b) Since $3600 = 60^2$ and $60 = 2^2 \times 3 \times 5$, then $3600 = (2^2 \times 3 \times 5)^2 = 2^4 \times 3^2 \times 5^2$.
Thus, each divisor of 3600 contains at most the prime factors 2, 3 and 5, and cannot contain any other prime factors.
Further, the prime factorization of each divisor of 3600 contains at most four factors of 2, two factors of 3, and two factors of 5.
Suppose that $k = 2^x \times 3^y \times 5^z$. Then $20 \times k = 2^2 \times 5 \times 2^x \times 3^y \times 5^z = 2^{x+2} \times 3^y \times 5^{z+1}$ is a divisor of $3600 = 2^4 \times 3^2 \times 5^2$ exactly when $0 \leq x \leq 2$, $0 \leq y \leq 2$, and $0 \leq z \leq 1$, for integers x, y, z.
As noted in part (a), $20 \times k = 2^{x+2} \times 3^y \times 5^{z+1}$ is a perfect square exactly when each of the exponents, $x + 2$, y, and $z + 1$ is even.
Since $0 \leq x \leq 2$, then $x + 2$ is even when $x = 0$ or $x = 2$.
Since $0 \leq y \leq 2$, then y is even when $y = 0$ or $y = 2$.
Since $0 \leq z \leq 1$, then $z + 1$ is even when $z = 1$.
There are 2 choices for x, 2 choices for y, and 1 choice for z, and so there are $2 \times 2 \times 1 = 4$ possible values of k.
When $(x, y, z) = (0, 0, 1)$, we get $k = 2^0 \times 3^0 \times 5^1 = 5$.
When $(x, y, z) = (0, 2, 1)$, we get $k = 2^0 \times 3^2 \times 5^1 = 45$.
When $(x, y, z) = (2, 0, 1)$, we get $k = 2^2 \times 3^0 \times 5^1 = 20$.
When $(x, y, z) = (2, 2, 1)$, we get $k = 2^2 \times 3^2 \times 5^1 = 180$.
The positive integers k satisfying the given conditions are 5, 45, 20, and 180.

(c) Since $2025 = 45^2$ and $45 = 3^2 \times 5$, then $2025 = (3^2 \times 5)^2 = 3^4 \times 5^2$.
Since a^2 and b^2 are perfect squares, and $a^2 \times b^2 \times c = 2025$, then a^2 and b^2 are each equal to perfect square divisors of 2025.
The perfect square divisors of $2025 = 3^4 \times 5^2$ are: 1, 3^2, 3^4, 5^2, $3^2 \times 5^2$, and $3^4 \times 5^2$.
We count the number of ordered triples of positive integers (a, b, c) by considering the following 2 cases: (1) At least one of a^2 or b^2 is equal to 1; (2) Both a^2 and b^2 are not equal to 1.

Case 1: At least one of a^2 or b^2 is equal to 1

Suppose that $a^2 = 1$. Then b^2 can be equal to each of the perfect square divisors of 2025 previously listed.
That is, b^2 can be equal to each of the 6 values: 1, 3^2, 3^4, 5^2, $3^2 \times 5^2$, and $3^4 \times 5^2$.
For each of these values of b^2, $c = \dfrac{2025}{a^2 \times b^2}$. For example, when $a^2 = 1$ and $b^2 = 1$, then $c = \dfrac{3^4 \times 5^2}{1 \times 1} = 3^4 \times 5^2$, and so $(a, b, c) = (1, 1, 3^4 \times 5^2)$ is a possible ordered triple.
When $a^2 = 1$ and $b^2 = 3^2$, then $c = \dfrac{3^4 \times 5^2}{1 \times 3^2} = 3^2 \times 5^2$, and so $(a, b, c) = (1, 3, 3^2 \times 5^2)$ is a possible ordered triple.
Continuing in this way with $a^2 = 1$, we get the following 6 ordered triples (a, b, c):

$$(1, 1, 3^4 \times 5^2), (1, 3, 3^2 \times 5^2), (1, 3^2, 5^2), (1, 5, 3^4), (1, 3 \times 5, 3^2), (1, 3^2 \times 5, 1)$$

For each of the ordered triples above, with the exception of $(1, 1, 3^4 \times 5^2)$, we get a new ordered triple by switching a and b.
Each of these 5 new ordered triples can be determined by letting $b^2 = 1$ and following the process above, and thus each satisfies the given conditions.
There are a total of $5 \times 2 + 1 = 11$ ordered triples in this case.

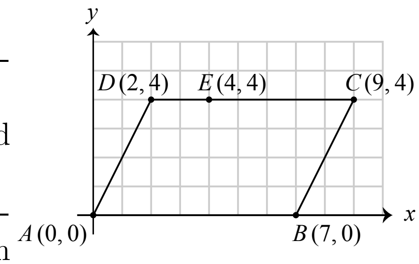
Case 2: Both a^2 and b^2 are not equal to 1

If one of a^2 or b^2 is equal to $3^4 \times 5^2$, then the other must be equal to 1 (since $2025 = 3^4 \times 5^2$).
In Case 2, both a^2 and b^2 are not equal to 1, and so each is also not equal to $3^4 \times 5^2$.
Removing these from our list of perfect square divisors, the possible values of a^2 and b^2 that remain are: 3^2, 3^4, 5^2, and $3^2 \times 5^2$.
If $a^2 = 3^2$, then b^2 can be equal to 3^2 or 5^2 or $3^2 \times 5^2$.
If $a^2 = 3^4$, then b^2 can be equal to 5^2.
If $a^2 = 5^2$, then b^2 can be equal to 3^2 or 3^4.
And finally, if $a^2 = 3^2 \times 5^2$, then b^2 can be equal to 3^2.
In each case, the value of c can be determined as it was in Case 1.
Doing so gives the following 7 ordered triples (a, b, c):

$$(3, 3, 5^2), (3, 5, 3^2), (3, 3 \times 5, 1), (3^2, 5, 1), (5, 3, 3^2), (5, 3^2, 1), (3 \times 5, 3, 1)$$

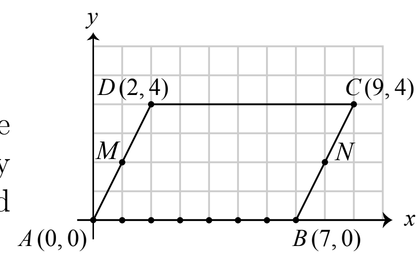
The number of ordered triples of positive integers (a, b, c) so that $a^2 \times b^2 \times c = 2025$ is $11 + 7 = 18$.

4. (a) We begin by placing $E(4,4)$ on CD, as shown.



To determine the area of each of the three triangles, consider the base of each to be AB.
Points C, D and E each have the same y-coordinate, 4, and so each lies on the horizontal line $y = 4$.
The height of each of the three triangles is the vertical distance between the line $y = 4$ and the x-axis (the line through A and B), which is 4.
Since $AB = 7$, then the area of each of the three triangles is $\frac{1}{2} \times 7 \times 4 = 14$.
The sum of the areas of $\triangle ABC$, $\triangle ABD$, and $\triangle ABE$ is $3 \times 14 = 42$.

(b) $\triangle CDG$ has non-zero area, so G cannot lie on CD.



On AB, there are 8 possible locations for G.
These are the points $(k, 0)$ for the integers $0 \leq k \leq 7$.
On AD, $(1, 2)$ is the only additional point for which the coordinates are both integers. Similarly, $(8, 2)$ is the only additional possibility for G on BC. The points $(1, 2)$ and $(8, 2)$ are labelled M and N respectively, as shown.
In total, there are 10 possibilities for the point G.

(c) Each such triangle has three vertices chosen from the 18 points with integer coordinates on the perimeter of $ABCD$, provided that the three vertices are not all on the same line.
These 18 points with integer coordinates are the 8 points on AB (including A and B), the 8 points on CD (including C and D), and the 2 points $M(1, 2)$ and $N(8, 2)$.
Each such triangle is described by exactly one of the following cases:

- the number of triangle vertices at M and N is 0
- the number of triangle vertices at M and N is 1
- the number of triangle vertices at M and N is 2

Case 1: the number of triangle vertices at M and N is 0

In this case, 2 vertices are on AB and 1 vertex is on CD, or vice versa.
Consider the triangles with 2 vertices on AB and 1 vertex on CD.
Consider the base of each such triangle to lie along AB.
There is 1 such triangle with vertices at $A(0, 0)$ and $B(7, 0)$, and thus has base length 7.
There are 2 such triangles with base length 6. One of these triangles has vertices at $A(0, 0)$ and $(6, 0)$, and the other has vertices at $(1, 0)$ and $B(7, 0)$.
Continuing in this way, there are 3 triangles with base length 5, 4 triangles with base length 4, 5 triangles with base length 3, 6 triangles with base length 2, and 7 triangles with base length 1.
Each of these triangles has height 4 since all points on CD are a vertical distance of 4 from any base that lies along AB (as in part (a)).
Suppose Q is one such point on CD having integer coordinates.
The sum of the areas of all triangles having 2 vertices on AB and 1 vertex at Q is

$$\frac{1}{2} \times 4 \times (1(7) + 2(6) + 3(5) + 4(4) + 5(3) + 6(2) + 7(1)) = \frac{1}{2} \times 4 \times 84 = 168$$

Also, for each of these bases, there are 8 possibilities for the third vertex that lies on CD.
These are the points $(k, 4)$ for integers $2 \leq k \leq 9$.
Thus, the sum of the areas of all triangles having 2 vertices on AB and 1 vertex on CD is $168 \times 8 = 1344$.

In a similar way, the sum of the areas of all triangles having 2 vertices on CD and 1 vertex on AB is also 1344, and so the sum of the areas of all triangles in Case 1 is $1344 \times 2 = 2688$.

Case 2: the number of triangle vertices at M and N is 1

This case can be divided into the following two subcases:
(i) 2 vertices are on AB (or 2 vertices are on CD), and 1 vertex is M or N
(ii) 1 vertex is on AB, 1 vertex is on CD, and 1 vertex is M or N

Subcase 2(i): 2 vertices on AB (or 2 vertices on CD), and 1 vertex is M or N

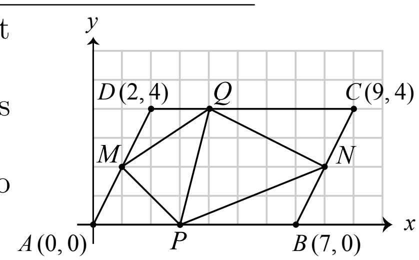
Consider the triangles with 2 vertices on AB and 1 vertex at either M or N.
Consider the base of each such triangle to lie along AB.
The numbers and lengths of these bases are the same as in Case 1.
Each of these triangles has height 2 since M and N are each a vertical distance of 2 from any base that lies along AB.
The sum of the areas of all triangles having 2 vertices on AB and 1 vertex at M is

$$\frac{1}{2} \times 2 \times (1(7) + 2(6) + 3(5) + 4(4) + 5(3) + 6(2) + 7(1)) = \frac{1}{2} \times 2 \times 84 = 84$$

Also, for each of these bases, the third vertex could also be N.
Thus, the sum of the areas of all triangles having 2 vertices on AB and 1 vertex at either M or N is $84 \times 2 = 168$.
In a similar way, the sum of the areas of all triangles having 2 vertices on CD and 1 vertex at either M or N is 168, and so the sum of the areas of all triangles in Subcase 2(i) is $168 \times 2 = 336$.

Subcase 2(ii): 1 vertex is on AB, 1 vertex is on CD, and 1 vertex is M or N

Consider two fixed points with integer coordinates, point P on AB, and point Q on CD.
In the diagram, $\triangle PMQ$ and $\triangle PNQ$ are two such triangles described by Subcase 2(ii).
The sum of the areas of $\triangle PMQ$ and $\triangle PNQ$ is equal to the sum of the areas of $\triangle PMN$ and $\triangle QMN$.



Consider the base of both $\triangle PMN$ and $\triangle QMN$ to be MN, which has length 7. Then the height of each triangle is 2.
Thus, the sum of the areas of $\triangle PMN$ and $\triangle QMN$ is $2 \times \frac{1}{2} \times 2 \times 7 = 14$.
There are 8 possible locations for P and 8 possible locations for Q, and thus $8 \times 8 = 64$ different pairs of triangles PMN and QMN whose areas have a sum of 14. (You should confirm for yourself that this is true even when P is $(0, 0)$ or $(7, 0)$ and/or when Q is $(2, 4)$ or $(9, 4)$.) The sum of the areas of all triangles in Subcase 2(ii) is $14 \times 64 = 896$.

Case 3: the number of triangle vertices at M and N is 2

In this case, 1 vertex is on AB (or on CD), 1 vertex is M, and 1 vertex is N.
Consider the triangles with vertices M, N, and 1 vertex on AB.
Consider the base of each such triangle to be $MN = 7$. Then each triangle has height 2.
Since there are 8 possible vertices on AB, then the sum of all such triangles is $\frac{1}{2} \times 2 \times 7 \times 8 = 56$. In a similar way, the sum of the areas of all triangles having vertices M, N, and 1 vertex on CD is also 56, and so the sum of the areas of all triangles in Case 3 is $56 \times 2 = 112$.

The sum of the areas of all triangles whose vertices have integer coordinates and lie on the perimeter of $ABCD$ is $2688 + 336 + 896 + 112 = 4032$.