The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2025 Fermat Contest

(Grade 11)

Wednesday, February 26, 2025
(in North America and South America)

Thursday, February 27, 2025
(outside of North America and South America)

Solutions

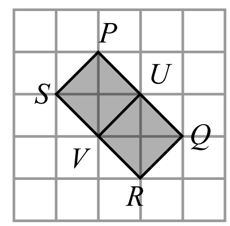
1. Evaluating following order of operations, we get $2 - 0 + 2 \times 5 = 2 - 0 + 10 = 12$.
   Answer: (D)

2. Reading from the graph, 20% of the students surveyed chose Friday.
   Since 3000 students were surveyed, the number of students who chose Friday as their favourite day of the week was $\frac{20}{100} \times 3000 = 20 \times 30 = 600$.
   Answer: (B)

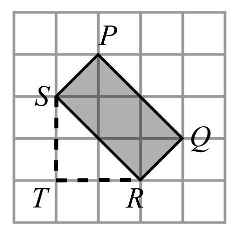
3. The integer values of x that satisfy the inequality $2 < x < 14$ are the integers between 3 and 13, inclusive, and so there are $13 - 3 + 1 = 11$ such integers.
   Answer: (E)

4. Alfonzo is paid \$14 of the \$50. Together, Rachel and Christophe are paid \$50 − \$14 = \$36.
   Rachel is paid twice what Christophe is paid, and so Rachel is paid $\frac{2}{3}$ of \$36 and Christophe is paid $\frac{1}{3}$ of \$36.
   Thus, Christophe is paid $\frac{1}{3} \times \$36 = \$12$.
   Answer: (B)

5. *Solution 1*
   We begin by placing U and V at the intersection of the gridlines shown.
   (You should confirm for yourself why U and V lie on PQ and RS, respectively.)
   Each of the line segments PU, UQ, QR, RV, VS, SP, and UV is a diagonal of a 1×1 square, and thus divides the square into two triangles having equal areas.
   Rectangle $PQRS$ contains 8 such triangles, each with area $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$, and so the area of $PQRS$ is $8 \times \frac{1}{2} = 4$.



   Solution 2
   We begin by constructing right-angled $\triangle STR$, as shown.
   By the Pythagorean Theorem, $SR^2 = ST^2 + TR^2 = 2^2 + 2^2 = 8$, and so $SR = \sqrt{8}$ (since $ST > 0$).
   Similarly, $QR^2 = 1^2 + 1^2 = 2$, and so $QR = \sqrt{2}$ (since $QR > 0$).
   The area of rectangle $PQRS$ is $SR \times QR = \sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$.



   Answer: (C)

6. Since $\sqrt{2025} = 45$, the next largest perfect square greater than 2025 is $46^2 = 2116$.
   Thus, the smallest possible value of n is $2116 - 2025 = 91$.
   Answer: (E)

7. The correct answer is 13. Why?
   Since the scoring areas 10 and 5 are each multiples of 5, then any combination of these two scores is also a multiple of 5.
   The closest multiple of 5 less than 13 is 10, and so to obtain a total score of 13, a minimum of three arrows must each score 1 point (since $13 - 10 = 3$).
   However, at least one arrow is needed to score 10 points, and so at least four arrows are needed to score 13 points.
   We confirm that each of the remaining answers is possible since $16 = 10 + 5 + 1$, $11 = 5 + 5 + 1$, $7 = 5 + 1 + 1$, and $20 = 10 + 5 + 5$.
   Answer: (C)

8. Since 15 integers have an average of 18, then the sum of these 15 integers is $15 \times 18 = 270$.
Since 5 of these integers have an average of 12, then the sum of these 5 integers is $5 \times 12 = 60$.
Thus the sum of the other 10 integers is $270 - 60 = 210$, and so their average is $\frac{210}{10} = 21$.

Answer: (B)

9. *Solution 1*
Factoring the left side of the equation $x^2 - y^2 = 72$ gives $(x - y)(x + y) = 72$.
Substituting $x - y = 12$ into this equation, we get $12(x + y) = 72$, and so $x + y = \frac{72}{12} = 6$.

Solution 2
Solving the system of equations, we substitute the second equation $x = 12 + y$ into the first equation $x^2 - y^2 = 72$ to get $(12 + y)^2 - y^2 = 72$.
Solving, we have $144 + 24y + y^2 - y^2 = 72$ or $24y = -72$, and so $y = \frac{-72}{24} = -3$.
Since $x = 12 + y = 12 - 3 = 9$, then $x + y = 9 - 3 = 6$.

Answer: (E)

10. There are 50 groups, and so $m + n = 50$. There are 186 students, and so $3m + 4n = 186$.
Solving the system of equations, we substitute the first equation $m = 50 - n$ into the second equation $3m + 4n = 186$ to get $3(50 - n) + 4n = 186$.
Solving, we have $150 - 3n + 4n = 186$ or $n = 186 - 150 = 36$.
Since $m = 50 - n = 50 - 36 = 14$, then $m - n = 14 - 36 = -22$.

Answer: (A)

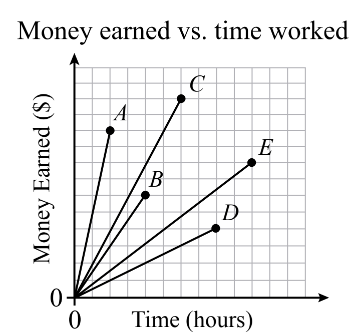
11. In the chart that follows, we number the lightbulbs 1 to 12 and represent those that are on with a 1 and those that are off with a 0.

	1	2	3	4	5	6	7	8	9	10	11	12
At the start (all lightbulbs are off):	0	0	0	0	0	0	0	0	0	0	0	0
After Angie flips every 2nd switch:	0	1	0	1	0	1	0	1	0	1	0	1
After Bilal flips every 3rd switch:	0	1	1	1	0	0	0	1	1	1	0	0
After Chenxhui flips every 4th switch:	0	1	1	0	0	0	0	0	1	1	0	1

At the end of the process, the lightbulbs numbered 2, 3, 9, 10, and 12 are on, and so 5 lightbulbs are on.

Answer: (A)

12. The vertical axis represents money earned (in dollars) and the horizontal axis represents time (in hours).
Thus, each employee's money earned per hour is the slope of the line segment from the origin $(0, 0)$ to the employee's letter.
The employee who was paid the most money per hour is the employee whose line segment has the greatest slope.
The line segment with the greatest slope is the steepest of the 5 line segments, and so Employee A was paid the most money per hour.
Alternately, we could add numbers to each of the axes and use them to determine the slope of each line, and thus the money earned per hour for each employee.



Answer: (A)

13. Since the equation is true for all real numbers x,it is true for $x = 0$.
Substituting $x = 0$, we get $(0 + 2)(0 + t) = 0^2 + b(0) + 12$ or $2t = 12$, and so $t = 6$.
The equation becomes $(x + 2)(x + 6) = x^2 + bx + 12$ or $x^2 + 8x + 12 = x^2 + bx + 12$, and so $b = 8$.

ANSWER: (B)

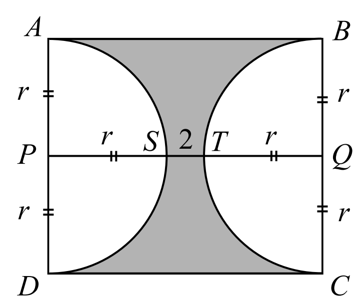
14. Each minute time moves forward, the volume of the substance doubles, and thus each minute time moves backward, the volume is halved.
The container was full at 9:20 a.m., and so it was half full at 9:19 a.m., and one-quarter full at 9:18 a.m.

ANSWER: (E)

15. Suppose the smallest number that is divisible by 12, 13, 14 and 15 is N.
Each number that is divisible by 12 has factors 3 and 4, since $3 \times 4 = 12$ and 3 and 4 have no factors in common.
Similarly, each number that is divisible by 14 has factors 2 and 7.
However, N must have a factor of 4 and thus it has a factor of 2.
N is divisible by $15 = 3 \times 5$ and so 5 is also a factor of N (3 was previously included).
The prime number 13 is also a factor of N.
The factors of N are 3, 4, 7, 5, and 13, and so $N = 3 \times 4 \times 7 \times 5 \times 13 = 5460$ (this is the lowest common multiple of 12, 13, 14 and 15).
The smallest five-digit number that is divisible by 12, 13, 14 and 15, is the smallest integer multiple of N that is greater than or equal to 10 000.
The smallest integer multiple of N that is greater than or equal to 10 000 is $2N = 2 \times 5460 = 10\,920$.
The hundreds digit of the smallest five-digit number that is divisible by 12, 13, 14 and 15 is 9.

ANSWER: (D)

16. Suppose the midpoints of AD and BC are P and Q, respectively.
Join P to Q and label the intersection of PQ with each circle S and T, as shown.
Since $AD = BC$, the semi-circles have equal diameters, and thus equal radii, r, and so $AP = PD = PS = BQ = QC = TQ = r$.
The shortest distance between the two semi-circles is $ST = 2$, and so $ABCD$ has dimensions $AD = 2r$ and $AB = PQ = 2r + 2$.
The area of $ABCD$ is $AD \times AB = 2r(2r + 2) = 224$.
Solving this equation, we get



$$\begin{aligned} 2r(2r+2) &= 224 \\ 4r^2 + 4r &= 224 \\ r^2 + r - 56 &= 0 \\ (r+8)(r-7) &= 0 \end{aligned}$$

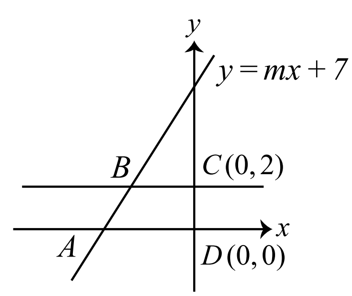
and so $r = 7$ (since $r > 0$). The area of the shaded region is the difference between the area of $ABCD$ and the combined areas of the two semi-circles, or $224 - \left(2 \cdot \frac{1}{2}\pi 7^2\right) \approx 70$.

ANSWER: (E)

17. The probability that Francesca wins her first game is equal to the probability that she plays either Dominique or Estella and wins, added to the probability that she plays any of the other 5 players and wins.
It is equally likely that Francesca plays her first game against any of the other 7 players, and so the probability that she plays against either Dominique or Estella is $\frac{2}{7}$.
The probability that she wins her first game when playing either of these two players is $\frac{2}{5}$, and so the probability that she plays her first match against Dominique or Estella and wins is $\frac{2}{7} \times \frac{2}{5} = \frac{4}{35}$.
The probability that Francesca plays her first game against any of the other 5 players is $\frac{5}{7}$.
The probability that she wins her first game when playing any of the other 5 players is $\frac{3}{4}$, and so the probability that she plays her first match against any of the other 5 players and wins is $\frac{5}{7} \times \frac{3}{4} = \frac{15}{28}$.
Thus, the probability that Francesca wins her first match is $\frac{4}{35} + \frac{15}{28} = \frac{16}{140} + \frac{75}{140} = \frac{91}{140} = \frac{13}{20}$.
ANSWER: (D)

18. When Arturo finishes the race, he is 200 m ahead of Morgan and 290 m ahead of Henri.
This means that Morgan runs 2000 m − 200 m = 1800 m in the same amount of time that it takes Henri to run 2000 m − 290 m = 1710 m.
If they each continue at their same speeds, Morgan runs the last $\frac{1800 \text{ m}}{9} = 200$ m of the race in the same amount of time that Henri runs $\frac{1710 \text{ m}}{9} = 190$ m.
Therefore, Morgan completes the 1800 m + 200 m = 2000 m race in the same amount of time that Henri runs 1710 m + 190 m = 1900 m. Morgan finishes the race 100 m ahead of Henri.
ANSWER: (B)

19. Let A and B be the points at which the line with equation $y = mx+7$ intersects the lines with equations $y = 0$ and $y = 2$, respectively. Also, suppose C has coordinates $(0, 2)$ and D has coordinates $(0, 0)$. The trapezoid in the problem is $ABCD$, as shown.
We can find the coordinates of A and B in terms of m.
To find the coordinates of A, we find the point of intersection of the line with equation $y = mx + 7$ and the line with equation $y = 0$. Setting $0 = mx + 7$, we get $x = -\frac{7}{m}$.



Note that the y-coordinate of A must be 0 since it is, by definition, on the line with equation $y = 0$.
Therefore, the coordinates of A are $\left(-\frac{7}{m}, 0\right)$. Similarly, the coordinates of B are $\left(-\frac{5}{m}, 2\right)$.
Trapezoid $ABCD$ has parallel bases AD and BC and height CD.
The two bases are horizontal and have lengths $AD = \frac{7}{m}$ and $BC = \frac{5}{m}$. The length of CD is 2.
Therefore, the area of $ABCD$ is $\frac{1}{2} \cdot CD \cdot (AD + BC) = \frac{1}{2} \cdot 2 \cdot \left(\frac{7}{m} + \frac{5}{m}\right) = \frac{12}{m}$.
It is given that the area of the trapezoid is 3, so we have $\frac{12}{m} = 3$, or $m = 4$.
ANSWER: (C)

20. There are no 2-digit integers with the given property. This is because the product of the digits of a 2-digit number is at most $9 \times 9 = 81 < 200$.

Observe that $200 = 2^3 \times 5^2$. We are essentially looking for products of 3, 4, or 5 digits that equal 200.
The only digit that is a multiple of 5 is 5 itself, so no matter how many digits m has, exactly two of its digits must be 5.
The number of 3-digit numbers m with the given property is 3, since if two of the digits are 5, the third must be 8. There are three 3-digit numbers with one digit equal to 8 and two digits equal to 5. They are 558, 585, and 855.

If m has four digits, then two digits are 5 and the other two digits have a product of 8.
The only ways to express 8 as a product of two integers is $8 = 2 \times 4$ and $8 = 1 \times 8$.
In either case, there are six ways to choose where the two digits equal to 5 go. They are

55_ _ 5_5_ 5_ _5 _55_ _5_5 _ _55

In each of these 6 cases, there are 2 ways to place the remaining digits. There are also two choices for the remaining two digits (2 and 4 or 1 and 8), so the number of 4-digit numbers with the given property is $6 \times 2 \times 2 = 24$.

Using a similar method of counting, we conclude that if m has 5 digits, then there are 10 ways to place the two digits that are equal to 5.
The remaining three digits must have a product of 8, so they must be 1, 1, and 8 or 1, 2, and 4 or 2, 2, and 2.
If the remaining digits are 1, 1, and 8, then there are 3 choices of where to place the remaining digits. This is because once the 8 is placed, the last two digits must be 1 (there is no choices).
If the remaining digits are 1, 2, and 4, then there are 6 ways to place the remaining digits. This is because there are 6 ways to order the digits 1, 2, and 4.
Finally, if the remaining digits are all 2, then there is only one way to place the digits.
Thus, there are $3 \times 10 + 6 \times 10 + 10 = 100$ 5-digit numbers with the given property.

The total number of integers m with $1 < m < 100\,000$ with the property that the product of the digits is 200 is $3 + 24 + 100 = 127$. The integer formed by the rightmost 2 digits of N is 27.

Answer: (B)

21. In a right-angled triangle, the hypotenuse is always the longest side. Hence, the hypotenuse has length c cm.
The other two lengths, a cm and b cm, are the lengths of the legs. Therefore, the area of the triangle is $\frac{1}{2}ab$ cm^2.
The area is given to be 54 cm^2, and so $\frac{1}{2}ab = 54$ or $ab = 108$.
Since a and b are integers, a and b must form a factor pair of 108. We are also given that $a < b$, so the only possibilities for the pair (a, b) are $(1, 108)$, $(2, 54)$, $(3, 36)$, $(4, 27)$, $(6, 18)$, and $(9, 12)$.
Using the Pythagorean Theorem, we must also have that $c = \sqrt{a^2 + b^2}$.

Observe that

$$\begin{aligned}\sqrt{1^2+108^2} &\approx 108.00463\\ \sqrt{2^2+54^2} &\approx 54.037\\ \sqrt{3^2+36^2} &\approx 36.1248\\ \sqrt{4^2+27^2} &\approx 27.2947\\ \sqrt{6^2+18^2} &\approx 18.9737\\ \sqrt{9^2+12^2} &= 15\end{aligned}$$

and so the only possible pair (a, b) that leads to an integer value of c is $a = 9$ and $b = 12$, so it must be true that $c = 15$.

Answer: 15

22. We will begin by making the substitution $A = 2^x$, $B = 2^y$, and $C = 3^z$.
With these substitutions, the three equations become

$$\begin{aligned} A + B + \frac{1}{3}C &= 2259 &(1)\\ AB + C &= 7073 &(2)\\ A + B + C &= 6633 &(3)\end{aligned}$$

Subtracting Equation (1) from Equation (3) gives $\frac{2}{3}C = 6633 - 2259 = 4374$.
Hence, $C = \frac{3}{2} \cdot 4374 = 6561$.
Substituting $C = 6561$ into Equation (2) gives $AB + 6561 = 7073$ or $AB = 512$.
Substituting $C = 6561$ into Equation (3) gives $A + B + 6561 = 6633$ or $A + B = 72$.
Multiplying both sides of $A + B = 72$ by A (since $A \neq 0$) gives $A^2 + AB = 72A$, into which we can substitute $AB = 512$ to get $A^2 + 512 = 72A$.
Rearranging gives $A^2 - 72A + 512 = 0$ which can be factored to get $(A - 64)(A - 8) = 0$.
If $A = 64$, then either $AB = 512$ or $A + B = 72$ can be used to deduce that $B = 8$, and if $A = 8$, then we could similarly deduce that $B = 64$.
Thus, we have that A and B are 8 and 64, but we cannot determine the order. Since $64 = 2^6$ and $8 = 2^3$, we conclude that x and y are 6 and 3, though we cannot determine with certainty which is which.
Returning to $C = 6561$, we have $3^z = 6561 = 3^8$ so $z = 8$.
Regardless of which of x and y is 8 and which is 64, we get that $xyz = 3 \times 6 \times 8 = 144$. The integer formed by the rightmost two digits of 144 is 44.

Answer: 44

23. Let x be the number of minutes that it would take the bigger ant to build an anthill alone and let y be the number of minutes that it would take the smaller ant to build an anthill alone.
Since the bigger ant can build an anthill in x minutes, the bigger ant builds $\frac{1}{x}$ anthills per minute.
Likewise, the smaller ant can build $\frac{1}{y}$ anthills per minute.
Thus, working together, the two ants build $\frac{1}{x} + \frac{1}{y}$ anthills per minute.

It is also given that it takes the two ants 24 minutes to build an anthill together, so this means they build $\frac{1}{24}$ anthills per minute working together.
Hence, we get the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{24}$.
Multiplying this equation through by $24xy$ gives $24y + 24x = xy$.
From the other given condition, we get $x = y - 14$, so we can substitute to get $24y + 24(y - 14) = (y - 14)y$.
Expanding and rearranging, this equation becomes $y^2 - 62y + 336 = 0$, which can be factored as $(y - 56)(y - 6) = 0$.
If $y = 6$, then $x = -8$, which does not make sense since x must be positive. Therefore, $y = 56$.

ANSWER: 56

24. Consider the function $g(x) = 59x$. Observe that $g(1) = 59$, $g(2) = 118$, and $g(3) = 177$, from which it follows that $f(1) = g(1)$, $f(2) = g(2)$, and $f(3) = g(3)$.
Now define a polynomial $h(x) = f(x) - g(x)$ so that, by construction, $x = 1$, $x = 2$, and $x = 3$ are roots of $h(x)$.
Algebraically, we also have that

$$h(x) = f(x) - g(x) = x^4 + px^3 + qx^2 + rx + s - 59x = x^4 + px^3 + qx^2 + (r - 59)x + s$$

but the important thing to notice here is that $h(x)$ is a degree four polynomial with a leading coefficient of 1.
Since we already know that $x = 1$, $x = 2$, and $x = 3$ are roots of $h(x)$, we conclude that

$$h(x) = (x - 1)(x - 2)(x - 3)k(x)$$

for some polynomial $k(x)$. However, because $h(x)$ has degree 4 and a leading coefficient of 1, it must be true that $k(x)$ has degree 1 and a leading coefficient of 1.
Thus, $h(x) = (x - 1)(x - 2)(x - 3)(x - a)$ for some real number a.
We now evaluate $h(x)$ at $x = 9$ and $x = -5$ to get

$$h(9) = (9 - 1)(9 - 2)(9 - 3)(9 - a) = 336(9 - a)$$
$$h(-5) = (-5 - 1)(-5 - 2)(-5 - 3)(-5 - a) = 336(5 + a)$$

Now using that $h(x) = f(x) - g(x)$ or $f(x) = h(x) + g(x)$, we can determine the value of $f(9) + f(-5)$ as follows:

$$\begin{aligned} f(9) + f(-5) &= h(9) + g(9) + h(-5) + g(-5) \\ &= 336(9 - a) + 9(59) + 336(5 + a) - 5(59) \\ &= 336(9) - 336a + 9(59) + 336(5) + 336a - 5(59) \\ &= 336(9 + 5) + 59(9 - 5) \\ &= 336(14) + 59(4) \\ &= 4940 \end{aligned}$$

Therefore, $T = 4940$, so the answer is $4 + 9 + 4 + 0 = 17$.

ANSWER: 17

25. We will take for granted the following fact: For integers a and b, b^2 is divisible by a^2 exactly when b is divisible by a.
Noting that $2025 = 45^2$, we can apply this to get that $(a_k)^2$ is divisible by 2025 exactly when a_k is divisible by 45.
Therefore, we really want to determine the number of k with $1 \le k \le 2025$ such that a_k is divisible by 45.

Since $45 = 5 \times 9$ and 5 and 9 share no common prime divisors, we get that a_k is divisible by 45 exactly when it is divisible by both 5 and 9.
Consider the following three claims about the integers in the sequence.
Claim 1: a_k is divisible by 5 exactly when k is 4 more than a multiple of 5.
Claim 2: a_k is divisible by 9 exactly when k is 10 more than a multiple of 12.
Claim 3: a_k is divisible by 45 exactly when k is 34 more than a multiple of 60.
The rest of the solution will be structured as follows:

- Prove Claim 3 assuming Claims 1 and 2.
- Answer the question using Claim 3.
- Prove Claims 1 and 2.

Proof of Claim 3 from Claims 1 and 2
One can check that 34 is the smallest positive integer that is both 4 more than a multiple of 5 and 10 more than a multiple of 12. By Claims 1 and 2, a_{34} is divisible by 45.
Claim 1 implies that the multiples of 5 in the sequence appear every 5 terms after a_{34}, so at a_k for $k = 39$, $k = 44$, and so on.
Claim 2 implies that the multiples of 9 in the sequence appear every 12 terms after a_{34}, so at a_k for $k = 46$, $k = 58$, and so on.
The least common multiple of 5 and 12 is 60, and so the integers in the sequence that are both multiples of 5 and 9 occur every 60 terms after a_{34}.
Therefore, the terms in the sequence that are multiples of 45 are exactly those of the form a_k where k is 34 more than a multiple of 60.

To answer the question, we count the non-negative integers m such that $1 \le 60m + 34 \le 2025$. Rearranging, we get $-33 \le 60m \le 1991$, and after dividing through by 34, we get $-\dfrac{33}{60} \le m \le \dfrac{1991}{60}$.
Since m is non-negative and $\dfrac{1991}{60} \approx 33.18$, we conclude that m can be any of the integers from 0 through 33 inclusive, of which there are 34.
The answer to the question is 34.

Finally we will prove Claims 1 and 2.
Proof of Claim 1: a_k is divisible by 5 exactly when k is 4 more than a multiple of 5
Suppose two consecutive terms in the sequence, a_n and a_{n+1}, are both multiples of 5. That is, suppose $a_n = 5x$ and $a_{n+1} = 5y$ for some integers x and y.
Then $a_{n+2} = a_n - a_{n+1} = 5x - 5y = 5(x - y)$ is a multiple of 5 as well.
Similarly, $a_{n+1} = a_{n-1} - a_n$, which can be rearranged to get $a_{n-1} = a_n + a_{n+1} = 5x + 5y = 5(x + y)$, which is also a multiple of 5.
We have shown that if two consecutive terms in the sequence are multiples of 5, then so is the term after them and the term before them.
This reasoning can be followed to conclude that *every* term in the sequence must be a multiple of 5.

Not every term in the sequence is a multiple of 5 (for example, $a_1 = 1$ is not a multiple of 5), so we conclude that there cannot be two consecutive terms in the sequence that are both multiples of 5.
By very similar reasoning, it can be shown that if both a_n and a_{n+2} are multiples of 5 (two terms that are two apart in the sequence), then every term in the sequence must be a multiple of 5.
Hence, we conclude that any two multiples of 5 in the sequence must have a minimum of two terms strictly between them.
Now consider two consecutive terms a_n and a_{n+1}. We can use the recursive definition to compute the next few integers in the sequence in terms of a_n and a_{n+1} as follows.

$$\begin{aligned} a_{n+2} &= -a_{n+1} + a_n \\ a_{n+3} &= -a_{n+2} + a_{n+1} \\ &= -(-a_{n+1} + a_n) + a_{n+1} \\ &= 2a_{n+1} - a_n \\ a_{n+4} &= -a_{n+3} + a_{n+2} \\ &= -(2a_{n+1} - a_n) - a_{n+1} + a_n \\ &= -3a_{n+1} + 2a_n \\ a_{n+5} &= -a_{n+4} + a_{n+3} \\ &= -(-3a_{n+1} + 2a_n) + 2a_{n+1} - a_n \\ &= 5a_{n+1} - 3a_n \end{aligned}$$

If we assume that a_n is a multiple of 5, then $a_{n+5} = 5a_{n+1} - 3a_n$ is also a multiple of 5. Moreover, there cannot be any multiples of 5 in the sequence between these two multiples of 5 because it would cause there to be two multiples of 5 at most 1 term apart.
Computing the first few terms of the sequence, we get $a_1 = 1$ and $a_2 = 3$, $a_3 = -3 + 1 = -2$ and $a_4 = -a_3 + a_2 = 2 + 3 = 5$, and so a_4 is divisible by 5.
From the above argument, a_{4+5m} is divisible by 5 for all positive integers m, and there are no other k for which a_k is divisible by 5.

Proof of Claim 2: a_k is divisible by 9 exactly when k is 10 more than a multiple of 12
By nearly identical reasoning to the proof of Claim 1, it can be shown that the terms in the sequence that are multiples of 3 occur every 4 terms. We will take this for granted in this proof.
Continuing to compute the first few terms of the sequence, we get

$$1, 3, -2, 5, -7, 12, -19, 31, -50, 81$$

and so we see that $81 = a_{10}$ is the first multiple of 9.
Continuing from where we left off in the proof of Claim 1, we can, assuming that a_n and a_{n+1} are consecutive terms in the sequence, compute the terms a_{n+2} through a_{n+12} in terms of a_n

and a_{n+1}. When doing this, we get the following.

$$\begin{aligned}
a_{n+2} &= -a_{n+1} + a_n \\
a_{n+3} &= 2a_{n+1} - a_n \\
a_{n+4} &= -3a_{n+1} + 2a_n \\
a_{n+5} &= 5a_{n+1} - 3a_n \\
a_{n+6} &= -8a_{n+1} + 5a_n \\
a_{n+7} &= 13a_{n+1} - 8a_n \\
a_{n+8} &= -21a_{n+1} + 13a_n \\
a_{n+9} &= 34a_{n+1} - 21a_n \\
a_{n+10} &= -55a_{n+1} + 34a_n \\
a_{n+11} &= 89a_{n+1} - 55a_n \\
a_{n+12} &= -144a_{n+1} + 89a_n
\end{aligned}$$

Now if a_n is a multiple of 9, then so is $a_{n+12} = -144a_{n+1} + 89a_n$ since 144 is a multiple of 9.
Given that a_{10} is a multiple of 9, this shows that a_{10+12m} is a multiple of 9 for all positive integers m, as well.
It remains to verify that there are no other multiples of 9 in the sequence.
Suppose there is some other multiple of 9. Then there is some n such that a_n is a multiple of 9 and a_{n+k} is a multiple of 9 for some k with $0 < k < 12$.
A multiple of 9 must be a multiple of 3, so from what was claimed at the beginning of the proof, we must have that either a_{n+4} or a_{n+8} is a multiple of 9.
If a_{n+4} is a multiple of 9, then $-3a_{n+1} + 2a_n$ is a multiple of 9. If both a_n and $-3a_{n+1} + 2a_n$ are multiples of 9, then so is $-3a_{n+1}$, which means a_{n+1} is a multiple of 3.
Thus, we must have that a_n and a_{n+1} are both multiples of 3. By reasoning used earlier, this would imply that every term in the sequence is a multiple of 3, which is not true.
Therefore, a_{n+4} is not a multiple of 9. A similar argument shows that a_{n+8} is not a multiple of 9.

ANSWER: 34