The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2025 Cayley Contest

(Grade 10)

Wednesday, February 26, 2025
(in North America and South America)

Thursday, February 27, 2025
(outside of North America and South America)

Solutions

1. Evaluating, we get $\dfrac{20+25}{25+20} = \dfrac{45}{45} = 1$.

   Answer: (B)

2. Mother bear gives a total of $4 \times 3 = 12$ fish to her bear cubs, and so she has $14 - 12 = 2$ fish left over.

   Answer: (B)

3. The 5 smaller shaded rectangles contribute $5 \times 4 = 20$ to the width of the large rectangle.
   The 4 smaller unshaded rectangles contribute $4 \times 8 = 32$ to the width of the large rectangle.
   The width of the large rectangle is $w = 20 + 32 = 52$.

   Answer: (C)

4. The average of the 4 expressions is 17, and so their sum is $4 \times 17 = 68$.
   Thus, $(n-3)+(n-1)+(n+1)+(n+3) = 68$ or $4n = 68$, and so $n = \dfrac{68}{4} = 17$.
   Can you see why the value of n is equal to the average?

   Answer: (C)

5. Since 25% of 600 is $\dfrac{25}{100} \times 600 = 25 \times 6 = 150$, then 150 people move to the empty theatre.
   The percentage of the seats now filled in the smaller theatre is $\dfrac{150}{200} \times 100\% = \dfrac{150}{2}\% = 75\%$.

   Answer: (D)

6. The area of $\triangle DEC$ is equal to the sum of the areas of $\triangle DEF$ and $\triangle DFC$, and so $t + 2t = 63$ or $3t = 63$ or $t = 21$. The area of $\triangle ABC$ is equal to the sum of the areas of $\triangle ADC$, $\triangle DBE$, $\triangle DEF$ and $\triangle DFC$, which is equal to $4t + t + t + 2t = 8t = 8(21) = 168$.

   Answer: (A)

7. Solving $50 - 2\sqrt{x} = 18$, we get $-2\sqrt{x} = 18 - 50$ or $-2\sqrt{x} = -32$ or $\sqrt{x} = 16$, and so $x = 16^2 = 256$.

   Answer: (D)

8. Since $AB = AD$, then $\triangle ABD$ is isosceles and $\angle ADB = \angle ABD = 80^\circ$.
   Since $\angle BDC$ is a straight angle, then $\angle ADC = 180^\circ - \angle ADB = 180^\circ - 80^\circ = 100^\circ$.
   $\triangle ADC$ is also isosceles (since $AD = CD$), and so $\angle CAD = \angle ACD$.
   The sum of the angles in $\triangle ADC$ is 180°, and so $\angle ADC + \angle CAD + \angle ACD = 180^\circ$ or $100^\circ + 2 \times \angle ACD = 180^\circ$ or $2 \times \angle ACD = 80^\circ$, and so the measure of $\angle ACD$ is 40°.

   Answer: (E)

9. If Teddy is able to build a stack with height 21 cm using exactly 4 blocks, then the average height (the vertical dimension) of each of the blocks is $\dfrac{21\text{ cm}}{4} = 5.25$ cm.
   However, the largest dimension of each block is 5 cm, and so it is not possible to build a stack with height 21 cm using 4 (or fewer) blocks.
   If Teddy builds a stack using 4 blocks positioned so that the vertical dimension of each is 4 cm, and adds 1 more block to the stack, positioned so that its vertical dimension is 5 cm, then the stack has height $4 \times 4\text{ cm} + 1 \times 5\text{ cm} = 21$ cm.
   The smallest number of blocks that Teddy can use to build a stack with height 21 cm is 5.
   Can you find two more ways that Teddy can build a 21 cm stack using exactly 5 blocks?

   Answer: (B)

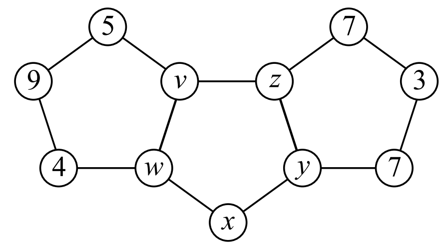
10. If $x = \frac{1}{4}$, then $-x = -\frac{1}{4}$, $x^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$, $3x = 3\left(\frac{1}{4}\right) = \frac{3}{4}$, and $\sqrt{x} = \sqrt{\frac{1}{4}} = \frac{1}{2}$.
The greatest of these numbers is $3x = \frac{3}{4}$.

Answer: (D)

11. Since $\sqrt{2025} = 45$, the next largest perfect square greater than 2025 is $46^2 = 2116$.
Thus, the smallest possible value of n is $2116 - 2025 = 91$.

Answer: (E)

12. We begin by labelling the missing numbers v, w, y, and z, as shown.
From the pentagon on the left, we get $4+9+5+v+w = 25$ or $18+v+w = 25$, and so $v+w = 7$.
From the pentagon on the right, we get $7+3+7+y+z = 25$ or $17+y+z = 25$, and so $y+z = 8$.
From the middle pentagon, we get $(v+w)+x+(y+z) = 25$ or $7+x+8 = 25$, and so $x = 25-7-8 = 10$.

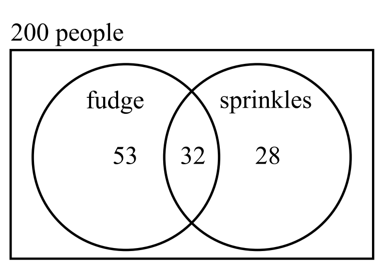


Answer: (E)

13. The probability that Farhan chooses a hat and a scarf having the same colour is equal to the probability that he chooses the white hat and the white scarf added to the probability that he chooses the blue hat and the blue scarf.
Since there are 2 hats, the probability that he chooses the blue hat is $\frac{1}{2}$.
Since there are 3 scarves, the probability that he chooses the blue scarf is $\frac{1}{3}$.
The probability that Farhan chooses both the blue hat and the blue scarf is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.
Similarly, the probability that Farhan chooses both the white hat and the white scarf is $\frac{1}{6}$.
The probability that Farhan chooses a hat and a scarf having the same colour is $\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$.

Answer: (A)

14. Of the 85 people who ordered fudge, 32 also ordered sprinkles, and so $85 - 32 = 53$ people ordered fudge without sprinkles.
Of the 60 people who ordered sprinkles, 32 also ordered fudge, and so $60 - 32 = 28$ people ordered sprinkles without fudge.
Therefore, 53 people ordered fudge only, 28 ordered sprinkles only, and 32 ordered both fudge and sprinkles.
Of the 200 people who bought ice cream, the number of people who ordered neither fudge nor sprinkles was $200 - 53 - 28 - 32 = 87$.
We can represent the given information in a Venn diagram, as shown below.



Answer: (C)

15. $\triangle ABC$ is right-angled at B, and so by the Pythagorean Theorem, we get $BC^2 = AC^2 - AB^2$ or $BC^2 = 40^2 - 20^2 = 1600 - 400 = 1200$, and so $BC = \sqrt{1200}$ (since $BC > 0$).
The radius of the semi-circle is $\frac{1}{2}BC = \frac{\sqrt{1200}}{2}$, and so its area is $\frac{1}{2}\pi\left(\frac{\sqrt{1200}}{2}\right)^2 = \frac{1}{2}\pi\left(\frac{1200}{4}\right)$, which when simplified is $\frac{1}{2}\pi(300) = 150\pi$.

ANSWER: (D)

16. Let the number of red marbles be $7n$, the number of yellow marbles be $3n$, and the number of green marbles be $5n$.
Note that the ratio of the number of red marbles to the number of yellow marbles to the number of green marbles is $7n : 3n : 5n = 7 : 3 : 5$, as is given.
The total number of marbles is 600, and so $7n + 3n + 5n = 600$ or $15n = 600$, and so $n = \frac{600}{15} = 40$.
Thus, the number of red marbles is $7n = 7(40) = 280$, the number of yellow marbles is $3n = 3(40) = 120$, and the number of green marbles is $5n = 5(40) = 200$.
If 20 marbles of each colour are removed, the number of red marbles is $280 - 20 = 260$, the number of yellow marbles is $120 - 20 = 100$, and the number of green marbles is $200 - 20 = 180$.
In this case, the new ratio of the number of red marbles to the number of yellow marbles to the number of green marbles is $260 : 100 : 180 = 26 : 10 : 18 = 13 : 5 : 9$.

ANSWER: (C)

17. Since $a^* = \frac{5}{a}$, then $100^* = \frac{5}{100}$, and so $(100^*)^* = \left(\frac{5}{100}\right)^*$.
Evaluating, $\left(\frac{5}{100}\right)^* = \frac{5}{\left(\frac{5}{100}\right)} = 5 \times \frac{100}{5} = 100$, and so $(100^*)^* = 100$.

ANSWER: (A)

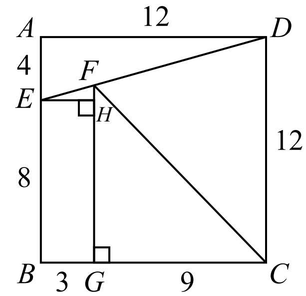
18. In step (i), Lavinia fills the 6 L bottle with water. In step (ii), she fills the 5 L bottle from the 6 L bottle, leaving the 6 L bottle with 6 L − 5 L = 1 L of water.
In step (iii), she empties the 5 L bottle and so the 5 L has no water and the 6 L bottle still contains 1 L.
In step (iv), she pours the 1 L of water from the 6 L bottle into the 5 L bottle.
After completing the sequence of four steps once, the 6L bottle contains no water and the 5 L bottle contains 1 L of water.
In the chart below, we continue to track the volume of water in each of the two bottles following each step. Each volume of water is measured in litres.

Step:	(i)	(ii)	(iii)	(iv)	(i)	(ii)	(iii)	(iv)	(i)	(ii)	(iii)	(iv)
6 L bottle	6	$6 - 5 = 1$	1	0	6	$6 - 4 = 2$	2	0	6	$6 - 3 = 3$	3	0
5 L bottle	0	5	0	1	1	5	0	2	2	5	0	3

After Lavinia completes the sequence of four steps a total of 3 times, the volume of water in the 5 L bottle is 3 L.

ANSWER: (D)

19. The area of $\triangle FGC$ is expressed as a fraction of the area of square $ABCD$, and so we begin by letting the side length of $ABCD$ be equal to 12, as shown, and its area $k = 12^2 = 144$.

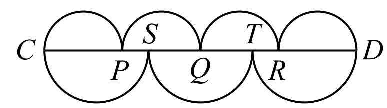


Since $AE = \frac{1}{3}AB = \frac{1}{3}(12) = 4$, then $EB = 12 - 4 = 8$.
Since $BG = \frac{1}{4}BC = \frac{1}{4}(12) = 3$, then $GC = 12 - 3 = 9$.
Next, we position point H on FG so that EH is perpendicular to FG, and so $EBGH$ is a rectangle with $GH = EB = 8$ and $EH = BG = 3$.
The slope of $ED = \dfrac{AE}{AD} = \dfrac{4}{12} = \dfrac{1}{3}$, and so the slope of EF is also equal to $\dfrac{1}{3}$.
The slope of $EF = \dfrac{FH}{EH}$ or $\dfrac{1}{3} = \dfrac{FH}{3}$, and so $FH = 3 \times \dfrac{1}{3} = 1$. Since $GH = 8$ and $FH = 1$, then $FG = 8 + 1 = 9$.
The area of $\triangle FGC$ is $\dfrac{1}{2}(GC)(FG) = \dfrac{1}{2}(9)(9) = \dfrac{81}{2}$.

At this point, we could substitute $k = 144$ into each of the given answers to determine which is equal to $\dfrac{81}{2}$. Doing so, we get $\dfrac{9}{32}k = \dfrac{9}{32}(144) = \dfrac{9}{2}(9) = \dfrac{81}{2}$, and so the answer is (A). Alternately, we could express $\dfrac{81}{2}$ in terms of k since $\dfrac{81}{2} = \dfrac{81}{2} \times \dfrac{k}{k} = \dfrac{81}{2} \times \dfrac{k}{144} = \dfrac{81}{2 \times 144}k = \dfrac{9}{2 \times 16}k = \dfrac{9}{32}k$.

Answer: (A)

20. Violet walks along 4 congruent semi-circles with diameters $CP = PQ = QR = RD = \dfrac{48 \text{ cm}}{4} = 12$ cm.



Thus, the total distance that Violet walks is $4 \times \dfrac{1}{2}\pi(12 \text{ cm}) = 24\pi$ cm.
Petunia walks along 3 congruent semi-circles with diameters $CS = ST = TD = \dfrac{48 \text{ cm}}{3} = 16$ cm.
Thus the total distance that Petunia walks is $3 \times \dfrac{1}{2}\pi(16 \text{ cm}) = 24\pi$ cm. Suppose that it takes Violet t seconds to walk from C to D. Then it takes Petunia $t - 12$ seconds to walk from C to D. Suppose that Violet travels at a constant speed of v cm/s. Then Petunia travels at a constant speed of $3v$ cm/s. This information is summarized in the table below.

	Distance (cm)	Time (s)	Speed (cm/s)
Violet	24π	t	v
Petunia	24π	$t - 12$	$3v$

Since distance is equal to the product of speed and time, and each ant travels the same distance, then $vt = 3v(t - 12)$.
Since $v > 0$, then we can divide both sides of the equation by v, and solving for t we get $t = 3(t - 12)$ or $t = 3t - 36$ or $36 = 2t$, and so $t = 18$.
Thus, it takes Violet 18 seconds to walk from C to D.

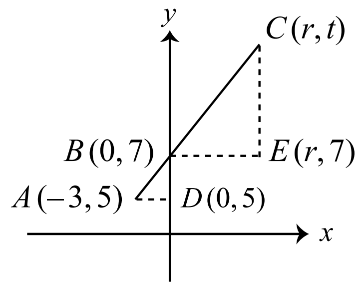
Answer: (B)

21. *Solution 1*

Plot the points A, B, and C as well as $D(0,5)$ and $E(r,7)$ so that $\triangle ABD$ and $\triangle BCE$ have right angles at D and E, respectively.

These two right-angled triangles each have a horizontal side and a vertical side, as shown. Since A, B, and C are all on the same line, AD is parallel to BE, and BD is parallel to CE, we must have that $\angle DAB = \angle EBC$ and $\angle DBA = \angle ECB$.

Hence, $\triangle ABD$ is similar to $\triangle BCE$.



Using common ratios, we get $\dfrac{BC}{AB} = \dfrac{EC}{DB} = \dfrac{BE}{AD}$.

Each of EC and DB is vertical, so their lengths are the difference between the y-coordinates of the two points.

Thus, $EC = t - 7$ and $DB = 7 - 5 = 2$.

Similarly, the lengths of BE and AD are each the different between the x-coordinates of the two points, giving $BE = r$ and $AD = 3$.

It is given that $BC = 4AB$, so $\dfrac{BC}{AB} = 4$. Therefore, $4 = \dfrac{EC}{DB} = \dfrac{t-7}{2}$ and $4 = \dfrac{BE}{AD} = \dfrac{r}{3}$.

Rearranging these two equations gives $8 = t - 7$ or $t = 15$ and $12 = r$, so $r + t = 27$.

Solution 2

Using the distance formula,

$$AB = \sqrt{(7-5)^2 + (0-(-3))^2} = \sqrt{4+9} = \sqrt{13}$$

and

$$BC = \sqrt{(t-7)^2 + (r-0)^2} = \sqrt{(t-7)^2 + r^2}$$

It is given that $BC = 4AB$, so $4\sqrt{13} = \sqrt{(t-7)^2 + r^2}$. Squaring both sides gives $16 \times 13 = (t-7)^2 + r^2$.

It is also given that A, B, and C are on a common line. This implies that the slope of the segment AB is the same as the slope of the segment BC.

These slopes are $\dfrac{7-5}{0-(-3)} = \dfrac{2}{3}$ and $\dfrac{t-7}{r-0} = \dfrac{t-7}{r}$, respectively.

Setting the computed slopes equal, we have $\dfrac{t-7}{r} = \dfrac{2}{3}$, or $t - 7 = \dfrac{2}{3}r$.

Substituting into $16 \times 13 = (t-7)^2 + r^2$, we get the following equivalent equations.

$$\begin{aligned} 16 \times 13 &= \left(\frac{2}{3}r\right)^2 + r^2 \\ 16 \times 13 &= \frac{4}{9}r^2 + r^2 \\ 16 \times 13 &= \frac{13}{9}r^2 \\ 16 \times 9 &= r^2 \\ \sqrt{16}\sqrt{9} &= \sqrt{r^2} \\ 12 &= r \end{aligned}$$

where the final equality is because r is assumed to be positive.

Therefore, we have $r = 12$, from which we get $t - 7 = \dfrac{2}{3} \times 12 = 8$, or $t = 15$.

The answer to the question is $r + t = 12 + 15 = 27$.

Answer: 27

22. We will count the Katende numbers that are at least 2401 and at most 2499, then we will count those that are at least 2500 and at most 2599.
Suppose $24AB$ is a Katende number where A and B are the tens and units digits of the integer.
Then there must be integers n and m such that $nm = 24$ and $n(m+1) = AB$.
Therefore, (n,m) must be a divisor pair of 24, so (n,m) is one of $(1,24)$, $(2,12)$, $(3,8)$, $(4,6)$, $(6,4)$, $(8,3)$, $(12,2)$, $(24,1)$.
If $n = 1$ and $m = 24$, then $n(m+1) = 25$, which gives the Katende number 2425.
Continuing in this way with the other seven divisor pairs of 24, we get the Katende numbers 2426, 2427, 2428, 2430, 2432, 2436, 2448.
There are a total of 8 Katende numbers with first two digits 24.

Using similar reasoning, assume that $25AB$ is a Katende number. Then there must be integers n and m such that $25 = nm$ and $AB = n(m+1)$.
The divisor pairs (n,m) of 25 are $(1,25)$, $(5,5)$, $(25,1)$.
These lead to the Katende numbers 2526, 2530, and 2550, for a total of 3.
Adding to the earlier total of 8 Katende numbers, we find that there are $8+3 = 11$ Katende numbers that are greater than 2400 and less than 2600.

Answer: 11

23. Let A, B, C, and D be the number of coins that Amr, Bai, Cindy, and Derek received, respectively. Then $A+B+C+D = N$. The three conditions given on how the coins were distributed correspond to the equations

$$A = \frac{1}{3}(B+C+D)$$
$$B = \frac{1}{5}(A+C+D)$$
$$C = \frac{1}{7}(A+B+D)$$

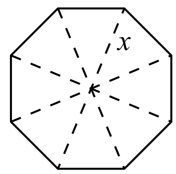
Scaling these three equations by 3, 5, and 7, respectively, we get

$$3A = B+C+D \qquad (1)$$
$$5B = A+C+D \qquad (2)$$
$$7C = A+B+D \qquad (3)$$

Subtracting Equation (1) from Equation (2) gives $5B-3A = (A+C+D)-(B+C+D)$ which can be simplified to $6B = 4A$ or $B = \frac{2}{3}A$.
Subtracting Equation (1) from Equation (3) gives $7C-3A = (A+B+D)-(B+C+D)$ which can be simplified to $8C = 4A$ or $C = \frac{1}{2}A$.
Substituting $B = \frac{2}{3}A$ and $C = \frac{1}{2}A$ into Equation (1) gives $3A = \frac{2}{3}A + \frac{1}{2}A + D$, or $D = \frac{11}{6}A$.
We have now expressed each of B, C, and D in terms of A, so we will substitute these expressions into the equation $A+B+C+D = N$ to get $A + \frac{2}{3}A + \frac{1}{2}A + \frac{11}{6}A = N$.
Simplifying the left side, we have $4A = N$. Since A and N are integers, this tells us that N must be a multiple of 4.
The largest multiple of 4 that is less than 100 is $N = 96$, so we guess that this is the answer.
If $N = 96$, then $4A = N$ implies $A = 24$, and using the equations for the other three variables in terms of A, we get $B = 16$, $C = 12$, and $D = 44$. One can check that these four integers satisfy the conditions given in the problem.

Answer: 96

24. If two side lengths of a triangle and the angle between them are known, then the area of the triangle can be computed. Specifically, if a and b are the lengths of two sides of a triangle and θ is the angle between those sides, then the area of the triangle is $\frac{1}{2}ab\sin\theta$.
We can use this to compute the areas of the octagon and the dodecagon.
Connect the centre of the octagon to each of the 8 vertices of the octagon, as shown.



This partitions the octagon into 8 triangles, each of which has two sides of length x and one side equal to the side-length of the octagon.
By side-side-side congruence, these eight triangles are congruent.
Suppose θ is the angle in each of these triangles made by the two sides of length x.
Then $8\theta = 360^\circ$ and so $\theta = 45^\circ$.
The area of the octagon is equal to the sum of the areas of the eight congruent triangles. Thus, the area of the octagon is

$$8 \times \frac{1}{2}x^2 \sin 45^\circ = 4x^2 \times \frac{1}{\sqrt{2}} = 4x^2 \times \frac{\sqrt{2}}{2} = 2\sqrt{2}x^2$$

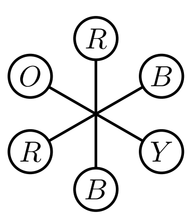
Using a similar construction, we find that the area of the dodecagon is

$$12 \times \frac{1}{2}y^2 \sin 30^\circ = 12 \times \frac{1}{2}y^2 \times \frac{1}{2} = 3y^2$$

Using that the radius of the circle is 3000, its area is $A = \pi(3000)^2$.
The areas of the octagon and dodecagon are each $\frac{1}{2}A$ or $2\pi(1500)^2$.
We can use this quantity to solve for each of x and y.
To solve for x, we have $2\pi(1500)^2 = 2\sqrt{2}x^2$ so $x^2 = \dfrac{\pi(1500)^2}{\sqrt{2}}$.
Taking square roots of both sides gives $x = 1500\sqrt{\dfrac{\pi}{\sqrt{2}}} \approx 2235.6752$.
To solve for y, we get $3y^2 = 2\pi(1500)^2$ or $y^2 = \dfrac{2\pi(1500)^2}{3}$, so $y = 1500\sqrt{\dfrac{2\pi}{3}} \approx 2170.8038$.
Hence, $x - y \approx 2235.6752 - 2170.8038 = 64.87$. Rounding to the nearest integer, the answer is 65.

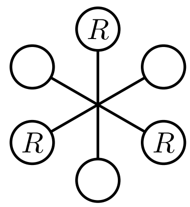
Answer: 65

25. Throughout the solution, we will use diagrams like the one below to represent a way to colour the flower. Each circle corresponds to a petal, and the capital letters R, O, Y, and B represent the colours red, orange, yellow, and blue, respectively. For example, the diagram representing the petals coloured red, blue, yellow, blue, red, orange starting at the top petal in clockwise order is



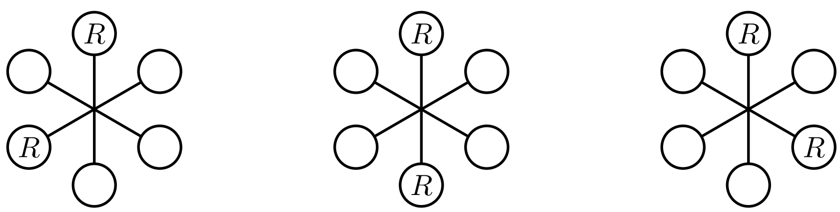
For now, we will count the number of ways to colour the petals according to the rules with the additional assumption that the top petal is red. Using this assumption, we will consider cases based on how many times the colour red is used.
Before reading on, you should convince yourself that it is not possible to colour four or more petals red without causing two neighbouring petals to be coloured red.

<u>*Case 1*</u>: Red is used three times.
Because two neighbouring petals must have different colours, there is only one choice of which three petals are coloured red

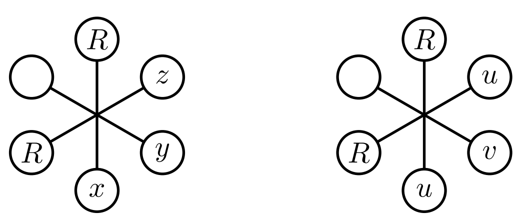


The rest of the petals can be coloured in any way using the other three colours, so there are $3\times3\times3 = 27$ ways to colour the petals in this case.

<u>*Case 2*</u>: Red is used two times.
There are three possibilities of which two petals are coloured red.



Consider the second of these configurations.
The two petals that neighbour the top red petal can each be coloured any of the three remaining colours.
Once these two (independent) choices are made, the remaining two petals can be independently coloured using one of exactly two possible colours.
Therefore, the colouring of the middle configuration can be completed in $3 \times 3 \times 2 \times 2 = 36$ different ways.
In the first configuration, the petal between the two red petals can be coloured in any of 3 ways, and this choice is independent of the way the other three petals are coloured.
The other three petals can either be coloured three different colours (x, y, and z), or they can be coloured using two different colours (u and v), as shown below:



If three colours are used, then there are 6 ways to choose them since we can choose x in any of 3 ways, then choose y in any of two ways, and then z is forced.
If two colours are used, then there are 3 ways to choose u and then 2 ways to choose v, for a total of 6 ways.
Thus, there are $6 + 6 = 12$ ways to colour the three petals (other than the one between the two red petals).
Since there are 3 ways to colour the petal between the red petals, there are $3 \times 12 = 36$ ways to complete the colouring for this configuration.

The third configuration can be coloured in 36 ways as well, so there are $36 + 36 + 36 = 108$ ways to colour the petals in this case.

Case 3: Red is used one time.
We can imagine colouring the petals one at a time in the clockwise direction starting from the red petal.
There will be 3 choices for the first petal since the only restriction is that we cannot use red.
The next petal has 2 options since it cannot be the same colour as the previous petal, but it also cannot be red.
Continuing in this way, there are 2 options for the 4^{th} petal, and 2 options for the 5^{th} petal.
The final petal also has two options since its two neighbours have two different colours. This is because we have not used red other than to colour the top petal.
Therefore, there are $3 \times 2 \times 2 \times 2 \times 2 = 48$ ways to colour the petals in this case.

From the three cases, we get a total of $27 + 108 + 48 = 183$ ways to colour the petals so that the top petal is red.
There are 4 choices for the colour of the top petal and each will lead to a count of 183.
Therefore, the number of ways to colour the petals according the the given rules is $4 \times 183 = 732$.
The last two digits of 732 make the integer 32, which is the answer to the question.

ANSWER: 32