The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
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2025
Canadian Team Mathematics Contest

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Solutions

Centre for Education in Mathematics and Computing Faculty and Staff / Personnel du Centre d'éducation en mathématiques et en informatique

Problems Committee / Comité des problèmes

Individual Problems

1. Since $m = 4$, then $(m+1)(m+2) = (4+1)(4+2) = 5 \times 6 = 30$.

   Answer: 30

2. The sum of the digits of n is $7 + A + 6 + 6 + 5 + B + 2 + A + 7 = 33 + 2A + B$, which we are told equals 46. Therefore $2A + B = 46 - 33 = 13$.
   The sum of the rightmost two digits of n is $A + 7$, which we are told equals 11. Therefore $A = 4$.
   Since $2(4) + B = 13$, we also have that $B = 5$, and therefore $A + B = 4 + 5 = 9$.

   Answer: 9

3. Let C be the capacity of the pot in litres. The given information says that $\frac{2}{3}C - 3 = \frac{1}{2}C$. Then $3 = \frac{2}{3}C - \frac{1}{2}C = \frac{1}{6}C$. Therefore $C = 3(6) = 18$ L.

   Answer: 18 L

4. *Solution 1*
   We observe the following:

$$\begin{aligned} 1+2+3+4+5 &= 15 \\ 2+3+4+5+6 &= 20 \\ 3+4+5+6+7 &= 25 \\ 4+5+6+7+8 &= 30 \end{aligned}$$

   Note that 15, 20, and 25 are not multiples of 6, and 30 is a multiple of 6.
   Since we require that all 5 consecutive integers be positive, we conclude that 30 is the smallest such multiple of 6.

   Solution 2
   Call our 5 consecutive integers $a-2, a-1, a, a+1, a+2$, for some integer $a \geq 3$. Then the sum of our 5 consecutive integers is $(a-2) + (a-1) + a + (a+1) + (a+2) = 5a$.
   Since we want $5a$ to be a positive multiple of 6, this means that $5a = 6b$ for some positive integer b.
   Since 5 and 6 have a greatest common divisor of 1, a must be a multiple of 6. The smallest positive multiple of 6 is 6 itself, and we note that the only solution with $a = 6$ is $b = 5$.
   Therefore the smallest positive multiple of 6 that can be expressed as a sum of 5 consecutive positive integers is $5a = 5(6) = 30$.

   Answer: 30

5. Since $\frac{AB}{BC} = \frac{2}{3}$, we may write $AB = 2x$ and $BC = 3x$ for some positive real number x.
   It follows that $AC = AB + BC = 2x + 3x = 5x$.
   Since $ABIH$ and $DEFJ$ are congruent squares, we calculate that

$$AG = AH + HG = AH + DE = 2x + 2x = 4x.$$

   Therefore the area of $ACEG$ is $2420 = (4x)(5x) = 20x^2$. Dividing by 20 tells us that $x^2 = 121$.
   Since $x > 0$ we conclude that $x = 11$.
   Finally, the perimeter of $ACEG$ is $2(AC) + 2(AG) = 2(5x) + 2(4x) = 18x = 18(11) = 198$.

ANSWER: 198

6. Let N be a possible number of stamps. The given conditions imply that there exist non-negative integers a, b, and c such that

$$N = 4a + 1$$
$$N = 5b + 2$$
$$N = 9c$$

We will find the smallest positive c for which this system has a solution; since $N = 9c$ then this will also determine the smallest positive integer N.
When $c = 1$ we have that $N = 9$, and we want the equations $9 = 4a + 1$ and $9 = 5b + 2$ to both have integer solutions. The first equation has a non-negative integer solution, but the second equation has no integer solution.
When $c = 2$ we have that $N = 18$, so we want $18 = 4a + 1$ and $18 = 5b + 2$ to both have integer solutions but neither of these have integer solutions.
We continue this search in the table below. Since we are starting with $c = 1$ and increasing c by 1 each time, our search is guaranteed to find the smallest possible value of N.

c	N	$N = 4a + 1$?	$N = 5b + 2$?
1	9	yes	no
2	18	no	no
3	27	no	yes
4	36	no	no
5	45	yes	no
6	54	no	no
7	63	no	no
8	72	no	yes
9	81	yes	no
10	90	no	no
11	99	no	no
12	108	no	no
13	117	yes	yes

Therefore, the smallest possible number of stamps is 117.

Editorial Comment: Imposing that $N = 4a + 1$ for some non-negative integer a is equivalent to imposing that N has remainder 1 when divided by 4. The **Chinese Remainder Theorem** tells us there is nothing special about the given remainders (0 when divided by 9, 1 when divided by 4, and 2 when divided by 5) since no two of $4, 5, 9$ have a common factor other than ± 1. That is, 0, 1, and 2 can be replaced with any possible remainders, and the system will still have a solution!

ANSWER: 117

7. Since our three integers are between 0 and 10, and are odd, they will each belong to the set $\{1, 3, 5, 7, 9\}$. Denote this set by S. Consider an ordered triple (a, b, c), where a, b, and c are integers from S, possibly not distinct. We wish to count how many such triples satisfy $a^2b^2c^2 = 2025$. Since a, b, and c are positive, this is equivalent to counting how many such

triples satisfy $abc = 45 = 5 \times 9$.

Since 5 is the only multiple of 5 in S, exactly one of a, b, c must be 5. Therefore the product of the other two numbers is 9, and the remaining two numbers are both 3, or one of them is 1 and the other of them is 9.

In summary, a, b, c are $5, 1, 9$ in some order, or are $5, 3, 3$ in some order. There are $3! = 6$ ways that a, b, c can be $1, 5, 9$ in some order, and there are 3 ways that a, b, c can be $3, 3, 5$ in some order. Therefore there are $6 + 3 = 9$ triples in which we can have $abc = 45$.

Since there are 5 choices for each of a, b, c, there are 5^3 different triples (a, b, c). Therefore, the probability that the product of the squares of the three numbers equals 2025 is $\dfrac{9}{5^3} = \dfrac{9}{125}$.

ANSWER: $\frac{9}{125}$

8. Note that $xy \neq 0$, $yz \neq 0$, and $xz \neq 0$. Thus we may take the reciprocals of each equation to get the system
$$\frac{4x + y}{xy} = \frac{10}{3} \qquad \frac{y + z}{yz} = \frac{10}{3} \qquad \frac{4x + z}{xz} = 14$$
Also, $x \neq 0$, $y \neq 0$, and $z \neq 0$ since $xy \neq 0$ and $xz \neq 0$. Let $a = \dfrac{1}{x}$, $b = \dfrac{1}{y}$, and $c = \dfrac{1}{z}$. Note that $\dfrac{4x + y}{xy} = \dfrac{4}{y} + \dfrac{1}{x} = 4b + a$. In terms of these new variables, the system therefore transforms to
$$4b + a = \frac{10}{3} \qquad c + b = \frac{10}{3} \qquad 4c + a = 14$$
This is a system of 3 linear equations in 3 unknowns.

The first equation, when solved for b, gives $b = \frac{1}{4}(\frac{10}{3} - a)$. The third equation, when solved for c, gives $c = \frac{1}{4}(14 - a)$. Substituting these expressions for b and c into the second equation yields $\frac{1}{4}(14 - a) + \frac{1}{4}(\frac{10}{3} - a) = \frac{10}{3}$. Solving for a yields that $a = 2$. Then the first equation gives $b = \frac{1}{3}$ and the third equation gives $c = 3$.

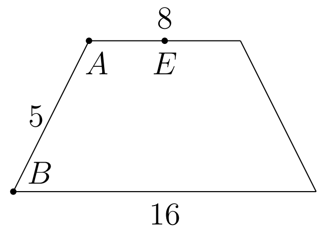
Therefore, $x = \frac{1}{2}$, $y = 3$, and $z = \frac{1}{3}$.

ANSWER: $(x, y, z) = \left(\frac{1}{2}, 3, \frac{1}{3}\right)$

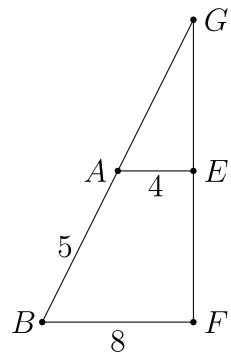
9. The circle with radius 5 has a circumference of 10π. The $72°$ angle given in the question tells us that the top circular face of the frustum will have circumference equal to $\dfrac{360 - 72}{360} = \dfrac{4}{5}$ times the circumference of the circle with circumference 10π, which equals 8π. Therefore the diameter of the top circular face is 8.

Similarly, the bottom circular face of the frustum has diameter $\dfrac{4}{5} \times (20\pi) = 16\pi$ and diameter of 16. Also, since $OA = 5$ and $OB = 10$, we conclude that $AB = 10 - 5 = 5$.

The diagram below is a cross section obtained by cutting the frustum with the vertical plane that contains points A and B, where we call the center of the top circular face E.



We divide the diagram in half using a vertical line through point E. Focusing on the left half after this division, we label point F as shown, and extend AB and EF to meet at point G:



Since $\angle BFG = 90°$, by the Pythagorean Theorem, we have that $EF = \sqrt{5^2 - (8-4)^2} = 3$. Since $\triangle BFG$ and $\triangle AEG$ are similar, and since $AE = \frac{1}{2}BF$, we have that $EG = \frac{1}{2}FG$, so

$$FG = 2(EG) = 2(FG - EF)$$

which can be rearranged to get $FG = 2(EF) = 6$. Using this and $EF = 3$ gives

$$EG = FG - EF = 6 - 3 = 3.$$

The volume of the frustum is equal to the volume of the cone with radius BF and height FG, minus the volume of the cone with radius AE and height EG.

Computing this quantity, we have

$$\begin{aligned}\frac{\pi}{3}(BF)^2(FG) - \frac{\pi}{3}(AE)^2(EG) &= \frac{\pi}{3}\left((BF)^2(FG) - (AE)^2(EG)\right)\\ &= \frac{\pi}{3}\left(8^2(6) - 4^2(3)\right)\\ &= \frac{\pi}{3}(3)(4^2)(2^2(2) - 1)\\ &= 16\pi(7)\\ &= 112\pi\end{aligned}$$

ANSWER: 112π

10. We will name the positions in an arrangement of 20 students as position 1 through position 20, from left to right. For some $1 \leq i \leq 19$, we say that an arrangement has a transition from position i to position $i+1$ if the students at these two positions have different coloured hats.

How many arrangements have a transition from position 1 to position 2? There are 8 options for the red-hatted students, 12 options for the green-hatted students, and 2 options for which goes in position 1 and which goes in position 2. There are then 18! ways to arrange the remaining 18 students (remember, we are assuming that the students are all different). Therefore, the number of sequences with a transition from position 1 to position 2 is

$$2 \times 8 \times 12 \times 18!$$

Similarly, this number is also the number of transitions that occur from spot i to spot $i+1$ for all $1 \leq i \leq 19$.

Therefore, the total number of transitions is $19 \times 2 \times 8 \times 12 \times 18!$. The average is computed to be

$$\frac{19 \times 2 \times 8 \times 12 \times 18!}{20!} = \frac{12 \times 8 \times 2}{20} = \frac{48}{5}$$

Answer: $\frac{48}{5}$

Team Problems

1. $\sqrt{25} = 5$ and $\sqrt{49} = 7$, so $\left(\sqrt{25} + \sqrt{49}\right)^2 = (5 + 7)^2 = 12^2 = 144$.

   Answer: 144

2. Converting to decimals, we want to count the odd integers between $\frac{13}{2} = 6.5$ and $\frac{37}{2} = 18.5$. These integers are 7, 9, 11, 13, 15, and 17, so there are 6.

   Answer: 6

3. Substituting $3w = x$ into $2x = y$ gives $2(3w) = y$ or $6w = y$.
   Substituting $6w = y$ into $5y = z$ gives $5(6w) = z$ or $30w = z$.
   Since $w \neq 0$, we can divide by w to get $30 = \frac{z}{w}$.

   Answer: 30

4. Using the definition of $5\Diamond x$, we get $5x + 5 + x = 41$ which is equivalent to $6x = 36$.
   Dividing both sides by 6 gives $x = 6$.

   Answer: 6

5. Suppose the shaded area is x. Then the area of each square is $3x$.
   The area of the entire figure is the sum of the areas of the two squares, but with the overlap subtracted since it should only be included once in total rather than once for each square.
   Thus, the area of the figure is $3x + 3x - x = 5x$, so the fraction of the figure that is shaded is $\frac{x}{5x} = \frac{1}{5}$.

   Answer: $\frac{1}{5}$

6. The first 180 km took a total of $\frac{180\,\text{km}}{45\,\text{km/hr}} = 4\,\text{hr}$.
   The second 180 km took a total of $\frac{180\,\text{km}}{90\,\text{km/hr}} = 2\,\text{hr}$.
   The trip took 6 hours, so the average speed for the trip was $\frac{360\,\text{km}}{6\,\text{hr}} = 60\,\text{km/hr}$.

   Answer: 60 km/hr

7. The given condition is that $(10 \times A + B)(10 \times C + D) = 100 \times B + 10 \times E + D$, which is equivalent to $100 \times A \times C + 10 \times (A \times D + B \times C) + B \times D = 100 \times B + 10 \times E + D$.

   We see that $B \times D$ and D must have the same remainder when divided by 10. By considering all the possibilities for B and D (there are 5(4)=20 such possibilities), we see that either $B = 1$, or $B = 3$ and $D = 5$.

   If $B = 1$, the identity becomes

   $$100 \times A \times C + 10 \times (A \times D + C) + D = 100 + 10 \times E + D$$

   Note that the right hand side is less than 200, while the left hand side is greater than 200, since $A \times C \geq 6$, since A, C cannot be 1.

   If $B = 3$ and $D = 5$, the identity becomes

   $$100 \times A \times C + 10 \times (5 \times A + 3 \times C) + 15 = 300 + 10 \times E + 5$$

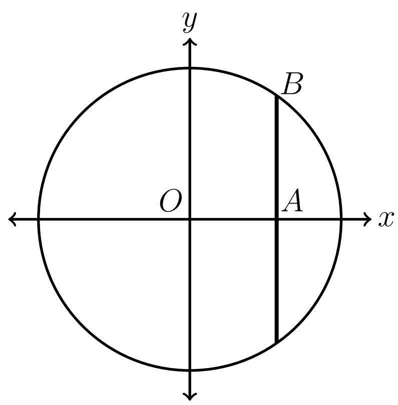
Note that the right hand side is less than 400. We see that the only way the left hand side can be less than 400 is if $A \times C = 2$, since the only possible values of $A \times C$ are 2, 4, and 8 (since 3 and 5 have already been used). Therefore, A and C are 1 and 2 in some order, and E is 4. The identity becomes

$$200 + 10 \times (5 \times A + 3 \times C) + 15 = 300 + 40 + 5$$

which means that $5 \times A + 3 \times C = 13$. Since $A = 1$ and $C = 2$ is not a solution to this, and $A = 2$ and $C = 1$ is a solution, we conclude that $A = 2$, $B = 3$, $C = 1$, $D = 5$, $E = 4$. Confirming, we indeed have that $23 \times 15 = 345$.

Answer: $A = 2$, $B = 3$, $C = 1$, $D = 5$, $E = 4$

8. By symmetry, the final answer is four times the length of a single chord. We will only work out the length of the chord determined by $x = 3$, and then multiply our final answer by 4.
   Let O be the centre of the circle, let A be the intersection of the chord with the x-axis, and let B be the upper intersection of the chord with the circle.



   Consider right-angled $\triangle OAB$: we have that $OB = 5$ and $OA = 3$.
   By the Pythagorean Theorem, $AB = \sqrt{5^2 - 3^2} = 4$.
   Therefore the total length of this chord is $2 \times 4 = 8$, and so the sum of all four length is $4 \times 8 = 32$.

   Answer: 32

9. The prime numbers less than 20, and greater than 1, are 2, 3, 5, 7, 11, 13, 17, 19. Since there are exactly 8 such prime numbers on that list, this means that the 8 chips must be labelled precisely with these numbers.

   The number of ways to select 2 of the 8 prime numbers, where we don't care about the order of selection, is $\binom{8}{2} = \frac{8 \times 7}{2} = 28$. In what follows, we treat pairs as being un-ordered.
   We count the number of ways that two of these chips have a units digit of 9 when the labels are multiplied.
   If one prime number has units digit of 1, the other prime number must have units digit of 9 - only the pair $11, 19$ satisfies these conditions.
   If one prime number has units digit 3, the other prime number must also have units digit 3 - only the pair $3, 13$ satisfies these conditions.
   If one prime number has units digit 7, the other prime number must also have units digit 7 - only the pair $7, 17$ satisfies these conditions.
   The prime number 2 cannot be used in any product, since multiples of 2 don't end in 9.

Similarly, the prime number 5 cannot be used in any product.
Therefore there are 3 different ways to choose 2 of the 8 prime number and get a units digit of 9.
Therefore the probability that the units digit of the product is 9 is $\frac{3}{28}$.

Answer: $\frac{3}{28}$

10. Denote the 5 digits as A, B, C, D, E. Then the condition we impose is that

$$A \times B \times C \times D \times E - 1 = A + B + C + D + E.$$

Let $L = A \times B \times C \times D \times E - 1$ and let $R = A + B + C + D + E$. Note that none of the digits can be 0, since then we would have $L = -1$ and $R \geq 0$, hence $L \neq R$.
We will find the smallest such number by using the most possible number of 1s, followed by the most possible number of 2s, etc., and then arranging these from left to right.

All 5 of the digits cannot be 1, since then we would have $L = 0$ and $R = 5$.

Could 4 of the digits be 1? Assume $A = B = C = D = 1$, and consider E as a variable digit. Then $L = E - 1$ and $R = 4 + E$. The condition $L = R$ is equivalent to the equation $E - 1 = 4 + E$, which has no digit solution. Therefore 4 of the digits cannot be 1.

Could 3 of the digits be 1? Assume $A = B = C = 1$, and consider D, E as variable digits. Then the constraint $L = R$ is equivalent to $D \times E - 1 = 3 + D + E$. This is equivalent to $(D - 1) \times (E - 1) = 5$. The only digit solution to this equation is for one digit to be 2 and for the other digit to be 6.

The above shows that the smallest number will use three 1s, one 2, and one 6. Therefore, the smallest possible integer is 11126.

Answer: 11126

11. Since $64 - 54 = 10$, for every 60 minutes that have elapsed on Clock A, the difference between Clock S and Clock F increases by 10 minutes. Initially, this difference is 0. When Clock S reads 7:20 p.m. and Clock F reads 8:10 p.m. there is a difference of 50 minutes between them. This means that $5(60) = 300$ minutes, or 5 hours, have elapsed on Clock A since all three clocks were set to the correct time.

Since $64 - 60 = 4$, after 5 hours have elapsed on Clock A, we have that $4(5) = 20$ additional minutes have elapsed on Clock F. Therefore, when Clock F reads 8:10 p.m., Clock A will read 7:50 p.m. Therefore, Joni set all three clocks to the correct time 5 hours before 7:50 p.m., giving us 2:50 p.m.

Answer: 2:50 p.m.

12. Suppose the first two numbers are a and b in that order. Then the sequence is

$$a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, 5a + 8b$$

and the sum of the numbers in the sequence is

$$a + b + (a + b) + (a + 2b) + (2a + 3b) + (3a + 5b) + (5a + 8b) = 13a + 20b$$

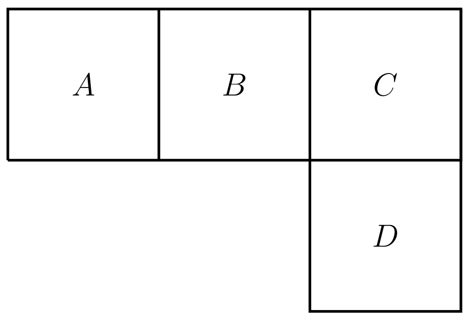
The seventh number in the sequence is given to be 97, so we get the equation $5a + 8b = 97$.
The sum of the seven numbers in the sequence is given to be 245, so we get the equation $13a + 20b = 245$.
Multiplying $5a + 8b = 97$ by 5 gives $25a + 40b = 485$, and multiplying $13a + 20b = 245$ by 2 gives $26a + 40b = 490$.
Subtracting these two equations gives $a = 5$. Substituting into $5a + 8b = 97$ gives $25 + 8b = 97$, so $8b = 72$, or $b = 9$.
Hence, $a = 5$ and $b = 9$, so the third number in the sequence is $a + b = 5 + 9 = 14$.

ANSWER: 14

13. Substituting $x = n$ gives $-16 = n^2 + 9n - n$, which is equivalent to $n^2 + 8n + 16 = 0$.
This equation factors as $(n + 4)^2 = 0$, so we conclude that $n = -4$.
Hence, $f(x) = x^2 + 9x - (-4) = x^2 + 9x + 4$, so $f(-2) = (-2)^2 + 9(-2) + 4 = 4 - 18 + 4 = -10$.

ANSWER: -10

14. We will refer to the four dice as die A, B, C, and D as shown in the diagram below.



Consider the face on die A that is touching a face on die B. Call this Face x for the moment.
This face does not have 5 or 3 dots on it, since other faces on die A have these numbers of dots.
Face x is also not opposite the faces with 5 and 3 dots on them, so Face x does not have $7 - 5 = 2$ or $7 - 3 = 4$ dots on it, either.
Therefore, Face x has either 1 dot or 6 dots on it.
The total number of dots on Face x and the face on die B that it touches is 8. Since there are at most 6 dots on any face of any die and $1 + 6 = 7 < 8$, Face x cannot have 1 dot on it.
Therefore, Face x has 6 dots on it.

Let Face y be the face on die B that is touching Face x. By the given condition, Face y has $8 - 6 = 2$ dots on it.
The face on die B that is opposite Face y has $7 - 2 = 5$ dots on it. Therefore, the face on die C that touches die B has $8 - 5 = 3$ dots on it.
Now consider die C. The face on top has 6 dots, the face touching die B has 3 dots, and so the face opposite the face touching die B has $7 - 3 = 4$ dots.
The face on the bottom (opposite the top face) has $7 - 6 = 1$ dot on it, so the face touching die D must have either 2 or 5 dots on it.
If it had 5 dots on it, then the face on die D that touches die C would have $8 - 5 = 3$ dots on it. However, the face with 3 dots on die D must be the bottom face, since it has to be opposite the side with 4 dots on it.
We conclude that the face on die C that touches die D must have 2 dots on it.
Therefore, the face on die D that touches die C has $8 - 2 = 6$ dots on it. This face is opposite the face marked with X, so the face marked with X has $7 - 6 = 1$ dot on it.

Answer: 1

15. We will use that an integer is divisible by 9 exactly when the sum of its digits is divisible by 9. Consider the Anderson number of $n = 999$. To compute the sum of its digits, we need to compute the sum of the digits of the integers 1, 2, 3, and so on up to 999.
It will be useful to think of all integers from 0 through 999 as three-digit numbers, with leading digits of 0 allowed. For example, we can think of 0 as 000, 5 as 005, and 45 as 045.
Since digits of 0 do not contribute to the sum of the digits, we can find the sum of the digits of the Anderson number of 999 by finding the sum of the digits of 000, 001, 002, 003, and so on to 999.
By allowing digits of 0, we get that each of the 10 digits, 0 through 9, occurs the same number of times among the three-digit numbers 000 through 999. There are a total of 1000 integers, each with 3 digits, so the total number of digits is 3000. There are 10 digits, so each digit occurs 300 times.
Therefore, the sum of the digits of the Anderson number of 999 is

$$300(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 300 \times 45 = 13500$$

Therefore, the sum of the digits of the Anderson number of 1000 is $13500+1+0+0+0 = 13501$.
Similarly, the sum of the digits of the Anderson number of 1001 is $13501+1+0+0+1 = 13503$.
Continuing in this way, we can compute the sum of the digits of the Anderson numbers of the next few four-digit integers. The results are summarized in the following table.

n	Sum of digits of Anderson number of n
1000	13501
1001	13503
1002	13506
1003	13510
1004	13515
1005	13521
1006	13528
1007	13536

One can check that $n = 1007$ is the smallest four-digit integer n for which the Anderson number of n is divisible by 9.

Answer: 1007

16. Suppose n has k digits and leading digit d. Then $n = 10^{k-1} \times d + m$ where m is the integer obtained by removing the leading digit from n.
Then we get the equation $10^{k-1}d + m = 57m$ or $10^{k-1}d = 56m$.
Since 56 is divisible by 7, the integer $10^{k-1}d$ must also be divisible by 7.
A power of 10 cannot be divisible by 7, so this forces d to be a multiple of 7. Since d is a digit, $d = 7$.
Using this and dividing both sides of $10^{k-1}d = 56m$ by 7 gives $10^{k-1} = 8m$.
Regardless of the integer m, $8m$ has at least three prime factors of 2. A power of 10 gets exactly one prime factor of 2 from each factor of 10, so this means $k - 1 \geq 3$, or $k \geq 4$.
If we take $k = 4$, then we have $10^3 = 8m$, and so $m = 125$.
The integer $n = 7125$ has the desired properties, and the argument above shows that it is the smallest.

ANSWER: 7125

17. The table below summarizes the number of segments used for each digit.

digit	0	1	2	3	4	5	6	7	8	9
# segments	6	2	5	5	4	5	6	3	7	6

The digit that uses the most segments is 8. If an integer has two digits equal to 8, then these two 8s take a total of $7 + 7 = 14$ segments.
The only digit that uses $17 - 14 = 3$ segments is 7. Thus the three integers 788, 878, and 887 are the only three integers that use exactly 17 segments and have two digits equal to 8.

If exactly one of the digits is 8, then the other two digits must use $17 - 7 = 10$ segments in total.
The only way to do this is for the two digits to use 4 and 6 segments or for them both to use 5 segments.
We first count the integers that have exactly one 8 and two digits that use 5 segments.
There are three choices for where to place the 8, and after the 8 is placed, there are three choices for the digit that uses 5 segments (either 2, 3, or 5) in each of the other two positions. This gives a total of $3 \times 3 \times 3 = 27$ integers in this case.
We next count the integers that have 8 as a digit, a digit that uses 4 segments, and a digit that uses 6 segments.
There are three digits that use 6 segments and 4 is the only digit that uses 4 segments.
There are three places that the 8 can go, then 2 places that the 4 can go, and three choices of which of 0, 6, and 9 to use (the position is determined after 8 and 4 are placed).
This gives a total of $3 \times 2 \times 3 = 18$ integers. However, this total counts the integers 084 and 048, which are not at least 100.
Therefore, there are $18 - 2 = 16$ integers in this case, and total of $27 + 16 = 43$ integers that use exactly 17 segments and have exactly one digit equal to 8.

The only way to express 17 as the sum of three nonnegative integers where each integer is less than 7 is $6 + 6 + 5$.
Therefore, if we do not use the digit 8, the integer must have two digits that use 6 segments and one digit that uses 5 segments.
There are three choices of where to place the digit that uses 5 segments and three choices of which digit to use. In all, there are $3 \times 3 = 9$ ways to select and place the digit that uses 5 segments.
Once this digit is placed, there are 3 ways to choose each digit that uses 6 segments, for a total of $9 \times 3 \times 3 = 81$ integers.
As in a previous case, we have counted some integers with leading digits equal to 0.
If 0 is the leading digit, then there are two choices of where to place the digit that uses 5 segments and three choices for that digit, for a total of $2 \times 3 = 6$ ways to place the digit that uses 5 segments.
There are three choices of which digit to place in the remaining position since it must be one of the three digits that use 6 segments. This gives a total of $6 \times 3 = 18$ integers that have a leading digit of 0.
Therefore, there are $81 - 18 = 63$ integers between 100 and 999 inclusive that use exactly 17 segments and do not have 8 as a digit.

Adding the totals together, there are $3 + 43 + 63 = 109$ integers.

ANSWER: 109

18. Since OPQ is a quarter circle, this means OB is a radius of the same circle and so $OB = 1$. As well, OB is a diagonal of square $OABC$, so the side length of the square is $\frac{OB}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.

Thus, the area of $OABC$ is $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

Quarter circle OAC has radius $OA = \frac{1}{\sqrt{2}}$, so its area is is $\frac{1}{4}\pi\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{8}$.

Thus, the area of the "largest" shaded region is $\frac{1}{2} - \frac{\pi}{8}$.

Suppose the total area of all shaded regions is x, and suppose the total area of all shaded regions except the largest is y.

We have just computed $x - y = \frac{1}{2} - \frac{\pi}{8}$, and we can use a notion of similarity to find another relationship between x and y.

More precisely, since there are infinitely many shaded regions, if we were to look separately at the part of the diagram that is within quarter circle OAC, it would look like a scaled version of the entire diagram.

Specifically, the following relationship holds:

$$\frac{y}{x} = \frac{\text{Area of } OAC}{\text{Area of } OPQ}$$

The area of OAC was computed as $\frac{\pi}{8}$ above, and the area of OPQ is $\frac{\pi}{4}$. Therefore

$$\frac{y}{x} = \frac{\frac{\pi}{8}}{\frac{\pi}{4}} = \frac{1}{2}$$

which can be rearranged to get $y = \frac{x}{2}$. This can be substituted into $x - y = \frac{1}{2} - \frac{\pi}{8}$ to get $x - \frac{x}{2} = \frac{1}{2} - \frac{\pi}{8}$ or $x = 1 - \frac{\pi}{4} = \frac{4-\pi}{4}$.

Answer: $\frac{4-\pi}{4}$

19. Adding the two equations gives $\sin^2 x + \cos^2 x + \cos^2 y - \sin^2 y = \frac{160}{144}$.

Using the Pythagorean identity, we have $\sin^2 x + \cos^2 x = 1$ and $\cos^2 y = 1 - \sin^2 y$, and we can substitute into the equation above to get $2 - 2\sin^2 y = \frac{160}{144}$.

Simplifying this equation gives $\sin^2 y = \frac{64}{144} = \frac{4}{9}$.

Since y is between 0° and 90°, $\sin y > 0$, and so we must have that $\sin y = \sqrt{\frac{4}{9}} = \frac{2}{3}$.

If we instead subtract the two given equations, we obtain $\sin^2 x - \cos^2 x + \cos^2 y + \sin^2 y = \frac{18}{144}$.

Substituting using the Pythagorean identity again gives $2\sin^2 x = \frac{18}{144}$ or $\sin^2 x = \frac{1}{16}$.

The angle x is also between 0° and 90°, so $\sin x > 0$, so $\sin x = \sqrt{\frac{1}{16}} = \frac{1}{4}$.

Answering the question, we have $\sin x + \sin y = \frac{1}{4} + \frac{2}{3} = \frac{11}{12}$.

Answer: $\frac{11}{12}$

20. Let $AB = x$, $AD = y$, and $AC = z$.
Applying the cosine law to $\triangle ADC$, we have $5^2 = y^2+z^2-2yz\cos\angle CAD$. Using $\cos\angle CAD = \dfrac{12}{13}$, we can simplify to $25 = y^2 + z^2 - \dfrac{24yz}{13}$.
Applying the Pythagorean Theorem to $\triangle ABC$, we have $z^2 = x^2+9^2$. Applying the Pythagorean Theorem to $\triangle ABD$, we have $y^2 = x^2 + 4^2$.
Since y and z are side lengths, they are positive, which means $z = \sqrt{x^2+81}$ and $y = \sqrt{x^2+16}$.
Substituting into the equation obtained earlier by using the cosine law, we get

$$25 = x^2 + 16 + x^2 + 81 - \frac{24\sqrt{x^2+16}\sqrt{x^2+81}}{13}$$

Rearranging slightly gives the equation $13(36 + x^2) = 12\sqrt{x^2+16}\sqrt{x^2+81}$.
Squaring both sides gives $13^2(36^2 + 72x^2 + x^4) = 12^2(x^4 + 97x^2 + 36^2)$, which can be rearranged to get

$$(13^2 - 12^2)x^4 + (72\times 13^2 - 97\times 12^2)x^2 + (13^2 - 12^2)36^2$$

Doing some calculator-free calculations, we have $13^2 - 12^2 = 169 - 144 = 25$, and

$$\begin{aligned} 72\times 13^2 - 97\times 12^2 &= 12(6\times 169 - 97\times 12) \\ &= 12(1014 - 1164) \\ &= -12\times 150 \\ &= -72\times 25 \end{aligned}$$

and so the quartic equation in x becomes $25x^4 - 72\times 25x^2 + 25\times 36^2 = 0$, and after dividing by 25, we get $x^4 - 72x^2 + 36^2 = 0$.
This quartic factors as $(x^2 - 36)^2 = 0$. Hence, $x^2 - 36 = 0$, so $x = \pm 6$, but x is a side length, so $x = 6$.

Answer: 6

21. Let $A = \log_y(x)$. Noting $A \neq 0$, by the change of base formula, we have

$$\frac{1}{A} = \frac{1}{\log_y(x)} = \frac{1}{\frac{\log_{10}(x)}{\log_{10}(y)}} = \frac{\log_{10}(y)}{\log_{10}(x)} = \log_x(y)$$

Using this, the equation $3\log_y(x) + 3\log_x(y) = 10$ becomes $3A + \dfrac{3}{A} = 10$. Multiplying through by A and rearranging gives $3A^2 - 10A + 3 = 0$.
Factoring, we have $(3A-1)(A-3) = 0$, and so $A = \log_y(x) = 3$ or $A = \log_y(x) = \dfrac{1}{3}$.
These two possibilities lead to $y^3 = x$ or $\sqrt[3]{y} = x$, the latter of which is equivalent to $x^3 = y$.
We have shown that either $x^3 = y$ or $y^3 = x$.
If $x^3 = y$, then we can multiply by x to get $x^4 = xy$, so $x^4 = 144$ by the first given equation.
Since x is positive, this means $x = \sqrt[4]{144} = \sqrt{12}$. Again using $xy = 144$, we get $y = \dfrac{144}{\sqrt{12}} = 12\sqrt{12}$, so $x + y = 12\sqrt{12} + \sqrt{12} = 13\sqrt{12} = 26\sqrt{3}$.
Nearly identical reasoning shows that if $y = x^3$, then $x + y = 26\sqrt{3}$.

Answer: $26\sqrt{3}$

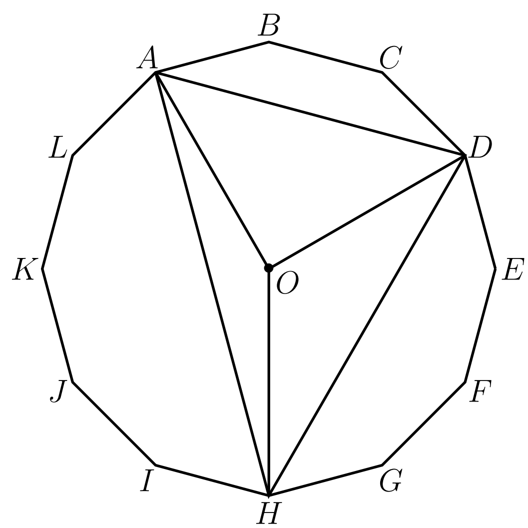
22. The condition that $f(c) = g(c) = 0$ means $c^2 - 2c - p = 0$ and $c^3 - 5c^2 - q = 0$.
Multiplying $c^2 - 2c - p = 0$ by c gives $c^3 - 2c^2 - pc = 0$.
Subtracting $c^3 - 5c^2 - q = 0$ from $c^3 - 2c^2 - pc = 0$ gives the equation $3c^2 - pc + q = 0$.
Multiplying $c^2 - 2c - p = 0$ by 3 gives $3c^2 - 6c - 3p = 0$.
Subtracting $3c^2 - pc + q = 0$ from $3c^2 - 6c - 3p = 0$ gives the equation $(p - 6)c - 3p - q = 0$ or $(p - 6)c = 3p + q$.

    If $p - 6$ is not 0, then $c = \dfrac{3p + q}{p - 6}$, but since p and q are both rational, the quantity on the right of this equation must be rational. We are given that c is irrational, so we conclude that $p - 6 = 0$, which is equivalent to $p = 6$.
It follows from $(p - 6)c = 3p + q$ that $3p + q = 0$ as well, so $q = -3p = -3(6) = -18$.

    The question is to find the real solutions to the equation $x^3 - 5x^2 - q = 0$ or $x^3 - 5x^2 + 18 = 0$. The cubic polynomial in this equation factors as $(x - 3)(x^2 - 2x - 6) = 0$ (one way to find this factorization is to use the Rational Roots Theorem). By the quadratic formula, the roots of $x^2 - 2x - 6 = 0$ are $1 \pm \sqrt{7}$, so the roots are 3, $1 + \sqrt{7}$, and $1 - \sqrt{7}$.

    Answer: $3, 1 + \sqrt{7}, 1 - \sqrt{7}$

23. A regular dodecagon can be circumscribed by a circle. Let O denote the center of this circle, which we also call the center of the dodecagon. Draw segments OA, OD, and OH, as shown in the diagram below:



    Note that the dodecagon is composed of the 12 congruent triangles $\triangle OAB$, $\triangle OBC$, …, and $\triangle OLA$. We have that

$$\angle AOB = \angle BOC = \cdots = \angle LOA = \frac{1}{12} 360^\circ = 30^\circ$$

    In particular, we have that $\angle AOD = 3(30^\circ) = 90^\circ$, $\angle DOH = 4(30^\circ) = 120^\circ$, and $\angle HOA = 150^\circ$.
Let r denote the radius of the dodecagon, by which we mean that $r = OA$.
We will repeatedly use that the area of a triangle is $\frac{1}{2}ab\sin\theta$ where a, b are side lengths of the triangle, and θ is the angle between the sides of lengths a and b.
The area of the full dodecagon is

$$\text{area}(ABCDEFGHIJKL) = 12 \times \frac{1}{2} r^2 \sin(30^\circ) = 3r^2$$

On the other hand, we have that

$$\begin{aligned}\text{area}(\Delta ADH) &= \text{area}(\Delta ADO) + \text{area}(\Delta DHO) + \text{area}(\Delta HAO)\\ &= \frac{1}{2}r^2\sin(90^\circ) + \frac{1}{2}r^2\sin(120^\circ) + \frac{1}{2}r^2\sin(150^\circ)\\ &= \frac{1}{2}r^2[\sin(90^\circ)+\sin(120^\circ)+\sin(150^\circ)]\\ &= \frac{1}{4}r^2(3+\sqrt{3})\end{aligned}$$

and we conclude that

$$\frac{\text{area}(\Delta ADH)}{\text{area}(ABCDEFGHIJKL)} = \frac{\frac{1}{4}r^2(3+\sqrt{3})}{3r^2} = \frac{3+\sqrt{3}}{12}$$

Answer: $\frac{3+\sqrt{3}}{12}$

24. Draw the line PD and consider the figures $ABCDP$ and $PAQD$.
The figure $ABCDP$ is a square pyramid with base $ABCD$, and $PAQD$ is a triangular pyramid with base AQD.
Observe that the entire pyramid $ABCDE$ is composed of these two pyramids along with the figure $EABPQ$.

Suppose, in general, that we have two pyramids, Pyramid 1 and Pyramid 2. Pyramid 1 has base area B_1 and height h_1 from its base, and that Pyramid 2 has base area B_2 and height h_2 from its base.
We can compute the ratio of the volumes of Pyramid 1 and Pyramid 2 in terms of the ratios of their base areas and heights. Specifically, we have

$$\begin{aligned}\frac{\text{volume of Pyramid 1}}{\text{volume of Pyramid 2}} &= \frac{\frac{1}{3}B_1h_1}{\frac{1}{3}B_2h_2}\\ &= \frac{B_1h_1}{B_2h_2}\end{aligned}$$

Let R be the point in $ABCD$ so that ER is perpendicular to the plane $ABCD$, and let T be the point in $ABCD$ such that PT is perpendicular to the plane $ABCD$ (think about why CTR is a line).
We have that $\triangle ERC$ is similar to $\triangle PTC$, and so $\frac{PT}{ER} = \frac{PC}{EC}$.
Using that $PC = EC - EP$ and that $\frac{EP}{EC} = \frac{1}{3}$, we get

$$\frac{PT}{ER} = \frac{PC}{EC} = \frac{EC-EP}{EC} = 1 - \frac{EP}{EC} = 1 - \frac{1}{3} = \frac{2}{3}$$

The pyramids $EABCD$ and $PABCD$ have a common base $ABCD$ (so their bases have the same area), and their heights are ER and PT, respectively. Using the result from earlier, we have

$$\frac{\text{volume of } PABCD}{\text{volume of } EABCD} = \frac{PT}{ER} = \frac{2}{3}$$

and since we are given that the volume of $EABCD$ is 28, we get that the volume of $PABCD$ is $28 \times \dfrac{2}{3} = \dfrac{56}{3}$.
Since $EABCD$ is a regular pyramid, the figure $EACD$ has half the volume of $EABCD$, so $EACD$ has a volume $\dfrac{28}{2} = 14$.

Figure $EACD$ is a pyramid with base $\triangle ADE$ and vertex C. As well, pyramid $PADQ$ has base $\triangle ADQ$, so the base of $PADQ$ is contained in the base of $EACD$, which is $\triangle ADE$.
Consider $\triangle ADE$ with base ED and $\triangle ADQ$ with base DQ.
These two triangles have the same height, h, from A. Therefore, the ratio of their areas is

$$\frac{\text{area } \triangle ADQ}{\text{area } \triangle ADE} = \frac{\frac{1}{2}(DQ)h}{\frac{1}{2}(ED)h} = \frac{DQ}{ED} = \frac{ED - EQ}{ED} = 1 - \frac{EQ}{ED} = 1 - \frac{1}{2} = \frac{1}{2}$$

We have now shown that pyramids $PADQ$ and $EACD$ have bases that are in the ratio $1 : 2$.
Using that $\dfrac{EP}{EC} = \dfrac{1}{3}$ in a way similar to earlier, we also get that their altitudes (from P to ADQ and C to ADE, respectively), are in the ratio $1 : 3$.
Using the fact about volumes of pyramids from earlier, we have that

$$\frac{\text{volume } PADQ}{\text{volume } EACD} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

and since the volume of $EACD$ is 14, we have that the volume of $PADQ$ is $\dfrac{14}{6} = \dfrac{7}{3}$.

We have now computed that the volume of $PABCD$ is $\dfrac{56}{3}$ and the volume of $PADQ$ is $\dfrac{7}{3}$, so the volume of figure $EABPQ$ is $28 - \dfrac{56}{3} - \dfrac{7}{3} = 7$.

Answer: 7

25. Throughout this solution, when we mention the moment, we mean the moment at which Ash tells a colleague that he already tested Program 6. We will consider separately the possibilities that Program 7 was written before the moment and that Program 7 has not yet been written.

If Program 7 was written before the moment, then all programs have been written. The number of possible ordered sequences of programs that Ash will test is equal to the number of possible lists of programs that remain to be tested.
Note that the list of programs that are ready to be tested is always in increasing order, so in this case, we really only need to count the number of possibilities for the set of programs that have not yet been tested (since each set can occur in only one possible order).
As it turns out, every possible subset of $\{1, 2, 3, 4, 5, 7\}$ could remain. For example, for the subset $\{3, 4, 7\}$ to be remaining, the following could have happened.

- Willow wrote Programs 1 and 2 before Ash started testing.
- Ash finished testing Program 2 and started testing Program 1 all before Willow finished writing Program 3.
- Willow finished programs 3, 4, 5, and 6 while Ash was testing Program 1.
- Ash tested Program 6 and started testing Program 5 while Willow was writing Program 7.
- Willow finished Program 7 and then Ash finished testing Program 5, at which time the moment started.

It is also possible for all programs to have been tested before the moment, and it is also possible that only Program 6 has been tested before the moment. For this to happen, it's possible that Willow wrote the first 6 programs before Ash arrived and finished program 7 before the moment, and the moment occurs immediately after Ash finished testing Program 6.

Thus, for each of Programs 1 through 7 (excluding 6, which is the only one we know has definitely been tested before the moment), it either has already been tested or it has not been tested yet. There are 6 programs in all and 2 options for each program, so there are $2^6 = 64$ possibilities for the sequence of programs to be tested after the moment.

We now consider the possibility that Program 7 was not completed before the moment.
Since Program 6 has been tested and the programs are written in order, this means Program 7 is the only one that still needs to be written after the moment.
Consider the list of programs waiting to be tested immediately after the moment. We are assuming that Program 7 is not in the list. By reasoning similar to that which was used in the previous case, the list could be any subset of Programs 1 through 5. Note, again, that for every subset, there is only one possible order.
As Ash begins testing programs, Program 7 might be finished at any time. Suppose, for example, that the list of programs ready to be tested immediately after the moment is 1, 3, 4. Then the sequence of programs tested in the afternoon could be any of 4, 3, 1, 7 or 4, 3, 7, 1 or 4, 7, 3, 1, for a total of 3 possibilities. Note that 7, 4, 3, 1 could be possible depending on exactly when Ash starts testing and when Program 7 is completed, but this sequence was accounted for in the previous case.
In general, if there are k programs in the list immediately after the moment, then there are k possible sequences, for each of the possible places in the sequence that Program 7 occurs, excluding the possibility that Program 7 is tested first since we have already counted these possibilities.
For each k from 0 through 5, there are $\binom{5}{k}$ subsets of $\{1, 2, 3, 4, 5\}$ of size k (and hence $\binom{5}{k}$ possible lists of programs of length k ready to be tested after the moment), and then k ways to place Program 7.
Thus, in this case we get

$$1\binom{5}{1} + 2\binom{5}{2} + 3\binom{5}{3} + 4\binom{5}{4} + 5\binom{5}{5} = 1(5) + 2(10) + 3(10) + 4(5) + 5(1) = 80$$

In total, there are $64 + 80 = 144$ possible sequences of programs that could be tested after the moment.

Answer: 144

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each relay are written as if the value of t is not initially known, and then t is substituted at the end.)

1. (a) *Solution 1*

   Listing the numbers, we have $3, 5, 6, 9, 10, 12, 15, 18, 20, 21, 24, 25$, for a total of 12 integers.

   Solution 2

   The inequality $1 \leq 3n \leq 25$ has 8 positive integer solutions, so there are 8 multiples of 3 between 1 and 25. Similarly, there are 5 multiples of 5 between 1 and 25. Note that only 15 is included in both these counts. Therefore our answer is $8 + 5 - 1 = 12$. [You can imagine that, for example, if 25 was replaced with 1000, this solution method would be much more efficient than the first solution.]

   (b) The given condition means that we have $t-x = y-t$, and therefore $x+y = 2t = 2(12) = 24$.

   (c) Setting equal the equations of the line and parabola gives us $3x + 10 = x^2 - 6x + t$, which is equivalent to $x^2 - 9x + (t - 10) = 0$. Therefore we have that

$$\begin{aligned} x &= \frac{9 \pm \sqrt{81 - 4(t-10)}}{2} \\ &= \frac{9 \pm \sqrt{121 - 4t}}{2} \\ &= \frac{9 \pm \sqrt{121 - 4(24)}}{2} \\ &= \frac{9 \pm \sqrt{25}}{2} \\ &= \frac{9 \pm 5}{2} \end{aligned}$$

   Therefore the smallest x value is $\dfrac{9-5}{2} = 2$.

   Answer: $(12, 24, 2)$

2. (a) There are 90 two-digit positive integers. Of these, 9 have repeated digits. Therefore there are $90 - 9 = 81$ two-digit positive integers with two different digits.

   (b) Let x denote the cost of one ounce of coffee and y denote the cost of one ounce of tea. It is given that $5x + 6y = t$ and $11x + 12y = 171$. We wish to determine y. Multiplying the first equation by 2 gives us that $10x + 12y = 2t$. Subtracting this from the second equation gives us that $x = 171 - 2t$. Using the first equation gives us that

$$y = \frac{t - 5x}{6} = \frac{t - 5(171 - 2t)}{6} = \frac{11t - 855}{6} = 6$$

   (c) We translate 2 units to the right by replacing x with $x - 2$. This gives us the parabola with equation

$$\begin{aligned} y &= (x-2)^2 + (m-t)(x-2) + 2m - 1 \\ &= x^2 + (m - t - 4)x + (4 + 2t - 1) \end{aligned}$$

Therefore the new y-intercept is $4 + 2t - 1 = 4 + 2(6) - 1 = 15$.
(Note that the value of m didn't matter. With knowledge of the rules of the event - you are trying to come up with a single number that doesn't depend on m - you could also have realized that the answer must not depend on m, and thus you could just assume $m = 0$.)

Answer: $(81, 6, 15)$

3. (a) A number is a multiple of 12 if and only if it is a multiple of 3 and a multiple of 4. Note that 2025 is a multiple of 3 since the sum of its digits is divisible by 3.
The next multiple of 3 is 2028.
Since the final two digits of 2028 are 28, which is divisible by 4, the answer is 2028.

(b) The volume of the green box is $10 \times 10 \times 13$. The given condition tells us that

$$\frac{t}{4} = \frac{n}{100} \times 10 \times 10 \times 13 = 13n$$

which simplifies to $n = \frac{t}{52} = \frac{2028}{52} = 39$.

(c) Since these 3 triangles are isosceles, and since 2 of their 3 lengths are radii of the circle, we conclude that each triangle has two occurrences of the angle $t°$, and that this occurrence does not occur at O.
Let A denote the third internal angle of each triangle, which we just noted must occur at O.
We have that $A + 2t = 180$, so $A = 180 - 2t$.
Also, we have that $3x + 3A = 360$, and therefore

$$x = 120 - A = 120 - (180 - 2t) = 2t - 60 = 2(39) - 60 = 18$$

Answer: $(2028, 39, 18)$