The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2025 Canadian Senior Mathematics Contest

Wednesday, November 12, 2025
(in North America and South America)

Thursday, November 13, 2025
(outside of North America and South America)

Solutions

Part A

1. On Tuesday, Wenfei eats 2 cookies at lunch, leaving him with $7 - 2 = 5$ cookies. After school, he buys 5 more cookies and has a total of $5 + 5 = 10$ cookies.

   On Wednesday, Wenfei eats 2 cookies at lunch, leaving him with $10 - 2 = 8$ cookies. After school, he buys 8 more cookies and has a total of $8 + 8 = 16$ cookies.

   On Thursday, Wenfei eats 2 cookies at lunch, leaving him with $16 - 2 = 14$ cookies. After school he buys 14 more cookies and has a total of $14 + 14 = 28$ cookies.

   On Friday, Wenfei eats 2 cookies at lunch, leaving him with $28 - 2 = 26$ cookies. After school he buys 26 more cookies and has a total of $26 + 26 = 52$ cookies.

   ANSWER: 52.

2. *Solution 1*
   If the equation is satisfied for all real numbers x, then it is satisfied when $x = 1$. When $x = 1$, $4x - 3 = 1$ and $16x - 12 = 4$, so we get $10 - k = 4$, so $k = 6$.

   Note: This solution would have worked in essentially the same way for any x value other than $x = \frac{3}{4}$.

   Solution 2
   Factoring the given expression, we get $(4x - 3)(10 - k) = 4(4x - 3)$.
   Rearranging gives $(4x - 3)(10 - k) - 4(4x - 3) = 0$, which can be further factored as $(4x - 3)(10 - k - 4) = (4x - 3)(6 - k) = 0$.

   This means that either $4x - 3 = 0$ or $6 - k = 0$. However, $4x - 3 = 0$ only when $x = \frac{3}{4}$, but the equation is true for all real numbers x.
   Therefore, it must be true that $6 - k = 0$ or $k = 6$.

   ANSWER: $k = 6$

3. *Solution 1*
   There are 6 possibilities for the first roll, and there are 6 possibilities for the second roll. Therefore, there are a total of $6 \times 6 = 36$ possibilities for the two rolls.

   If the first roll is 1, then there are 5 possibilities for the second roll that are greater than the first roll. They are 2, 3, 4, 5, and 6.
   If the first roll is 2, then there are 4 possibilities for the second roll that are greater than the first roll.
   In the table below, the first column contains the possible values for the first roll, the second column contains the possibilities for the second roll that are greater than the first roll, and the third column is the number of possibilities for the second roll that are greater than the first roll.

First roll	Possible second rolls	# of second rolls
1	2, 3, 4, 5, 6	5
2	3, 4, 5, 6	4
3	4, 5, 6	3
4	5, 6	2
5	6	1
6		0

There are $5+4+3+2+1+0=15$ possible pairs of rolls where the second roll is greater than the first roll. Since there are $6 \times 6 = 36$ possible pairs of rolls, the probability that the second roll is greater than the first roll is $\frac{15}{36} = \frac{5}{12}$.

Solution 2

The probability that the second roll matches the first roll is $\frac{1}{6}$.

The probability that the two rolls are different is $1 - \frac{1}{6} = \frac{5}{6}$.

For every pair of different integers from 1 through 6, there are two ways that these two rolls can appear: either the smaller number appears first, or the larger number appears first.

Therefore, if the two numbers are different, then there is a probability of $\frac{1}{2}$ that the smaller number appears first.

The probability that the second number is greater than the first number is $\frac{1}{2} \times \frac{5}{6} = \frac{5}{12}$.

Answer: $\frac{5}{12}$

4. Since x and y are perfect squares, we can write them as $x = n^2$ and $y = m^2$ for some non-negative integers n and m. The given equation can be rewritten as

$$\begin{aligned} x - y &= 35 \\ n^2 - m^2 &= 35 \\ (n+m)(n-m) &= 35 \end{aligned}$$

Since n and m are non-negative integers with a non-zero product, $n+m$ is a positive integer, and therefore $n-m$ is also a positive integer.

That n and m are non-negative implies $n+m \geq n-m$, and so $(n+m, n-m)$ must be a positive divisor pair of 35 with $n-m$ equal to the smaller of the two divisors.

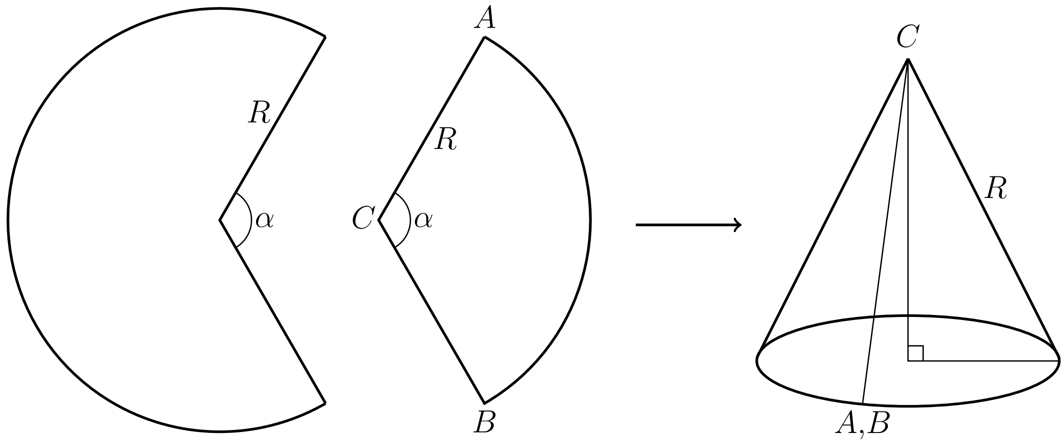
By considering positive divisors of 35, the possibilities for the pair $(n+m, n-m)$ are $(35, 1)$ and $(7, 5)$, so there are two possibilities: $n-m=1$ and $n+m=35$, or $n-m=5$ and $n+m=7$.

In the first case, adding the two equations gives $2n = 36$, which leads to $n = 18$ and $m = 17$. This case gives $x = 18^2 = 324$ and $y = 17^2 = 289$, so $x + y = 324 + 289 = 613$.

In the second case, adding the two equations gives $2n = 12$ which leads to $n = 6$ and $m = 1$. As a result, $x = 6^2 = 36$ and $y = 1^2 = 1$, so $x + y = 36 + 1 = 37$.

Answer: 613 and 37

5. We will first find a general formula for the volume of a cone created using a sector of angle α in degrees and a radius of R. The construction of the cone is illustrated in the diagram below.



The circumference of the base of the cone is equal to the length of the arc of the sector. The arc length of the sector is $\frac{\alpha}{360}$ of the total circumference of the circle, which is $\frac{\alpha}{360}(2\pi R) = \frac{\alpha\pi R}{180}$. Therefore, the radius of the base of the cone is $\frac{1}{2\pi} \times \frac{\alpha\pi R}{180} = \frac{\alpha R}{360}$.

The centre of the base of the cone, the vertex of the cone, and any point on the circumference of the base of the cone form a right-angled triangle. The legs of this triangle are the radius of the base and the height of the cone, and the hypotenuse equal to the slant length of the cone, which is R. Therefore, we have the following equivalent equations.

$$\begin{aligned}\left(\frac{\alpha R}{360}\right)^2 + h^2 &= R^2 \\ h^2 &= R^2\left(1 - \frac{\alpha^2}{360^2}\right) \\ h^2 &= R^2\frac{360^2 - \alpha^2}{360^2} \\ h &= \frac{R}{360}\sqrt{360^2 - \alpha^2}\end{aligned}$$

where in the last line, we use that h and R are positive, and that $360^2 > \alpha^2$ (since $360 > \alpha$).

We now have expressions for the radius and height of the cone in terms of the radius of the circle and the angle of the sector. Using the formula for the volume of a cone, we have

$$V = \frac{1}{3}\pi\left(\frac{\alpha R}{360}\right)^2 \frac{R}{360}\sqrt{360^2 - \alpha^2} = \frac{\pi\alpha^2 R^3\sqrt{360^2 - \alpha^2}}{3 \times 360^3}$$

In our case, we have $R = 1$, and we are interested in the value of θ so that $\alpha = \theta$ and $\alpha = 2\theta$ give rise to the same volume. Thus, we have the following equivalent equations

$$\begin{aligned}\frac{\pi\theta^2(1)^3\sqrt{360^2 - \theta^2}}{3 \times 360^3} &= \frac{\pi(2\theta)^2(1)^3\sqrt{360^2 - (2\theta)^2}}{3 \times 360^3} \\ \theta^2\sqrt{360^2 - \theta^2} &= (2\theta)^2\sqrt{360^2 - (2\theta)^2} \\ \sqrt{360^2 - \theta^2} &= 4\sqrt{360^2 - 4\theta^2} \qquad \text{(since } \theta \neq 0\text{)}\end{aligned}$$

and so we can now square both sides to get $360^2 - \theta^2 = 16(360^2 - 4\theta^2)$. Rearranging, we get $63\theta^2 = 15(360^2)$ or $\theta^2 = \frac{5}{21}(360)^2$. Taking square root gives $\theta = 360\sqrt{\frac{5}{21}}$.

Answer: $\theta = 360\sqrt{\frac{5}{21}}$

6. *Solution 1*

The number of 8-digit integers is 9×10^7 since in an 8-digit integer there are 9 choices for the leading digit (1 through 9) and 10 choices for each other digit (0 through 9). When the 0s are erased, each of these 9×10^7 integers contributes at least one integer to the new total. The challenge is to count how many additional integers it contributes.

Suppose an integer contributes at least two integers. Then it must have a digit of 0 followed immediately (to the right) by a non-zero digit. To build such an integer, there are 9 choices for the leading digit and 9 choices for the non-zero digit d that will go immediately after the 0. This pair, $0d$, can go in the second and third, third and fourth, and so on to the seventh and eighth positions, for a total of 6 possibilities. The remaining $8 - 3 = 5$ digits can be anything. This gives a total of $9^2 \times 6 \times 10^5$.

This seems like an overcount since, for example, the integers 90999099 is identified twice: once when the 09 is placed in the second and third positions, and once when the 09 is placed in the sixth and seventh position.

Notice that the integer 90999099 will contribute 3 integers to the new list. One of these is accounted for in the total of 9×10^7, and the other 2 appear precisely because there are two instances of a 0 followed by a non-zero digit. Thus, the number of times that this integer is counted by the total of $9^2 \times 6 \times 10^5$ from the previous paragraph is exactly the number of extra integers contributed by this integer to the final list.

In general, if there are k occurrences of a 0 followed immediately to the right by a non-zero digit, then the integer is counted k times in $9^2 \times 6 \times 10^5$, and it contributes an extra k integers. Therefore the overcount of $9^2 \times 6 \times 10^5$ exactly counts the number of additional integers contributed by erasing 0s.

Therefore, the answer is

$$9 \times 10^7 + 9^2 \times 6 \times 10^5 = 10^5(900 + 81 \times 6) = 10^5(900 + 486) = 138\,600\,000$$

Solution 2

Let S_n be the number of integers in the eventual list after erasing 0s, starting with all of the integer with n digits. Thus, we are looking for S_8.

For a positive integer n, let $T(N)$ be the number of integers contributed by N when zeros are erased. For example, if N has no digit equal to 0, then $T(N) = 1$. As another example, $T(1\,230\,456) = 2$ because it contributes the integers 123 and 456.

With these choices of notation, S_n is the sum of the values of $T(N)$ as N ranges over all n-digit positive integers.

Thus,

$$\begin{aligned} S_1 &= T(1) + T(2) + T(3) + T(4) + T(5) + T(6) + T(7) + T(8) + T(9) \\ &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\ &= 9 \end{aligned}$$

Also, $S_2 = 90$ since every 0 in a 2-digit integer must be the rightmost digit, so erasing it does not generate extra integers.

Consider S_3. There are 900 three-digit integers to start. In this case, the only way that erasing a zero can result in an extra integer is if the integer is of the form $x0y$ where x and y are non-zero. There are 81 such integers. For a 3-digit integer N, $T(N) = 2$ if N is of the form $x0y$ with x and y nonzero, and $T(N) = 1$ otherwise. As noted, there are 81 integers N with $T(N) = 2$, so $S_3 = 900 + 81 = 981$.

Consider S_4. There are 9000 four-digit integers to start. Integers that generate at least one more integer when split are of the form $x00y$, $xy0z$, or $x0ya$ where x, y and z are non-zero and a might be 0.

If $N = x00y$, then $T(N) = 2$. Another way to think about this is that $T(N) = T(x00y) = 1 + T(y)$. Therefore, each $x00y$ generates an additional $T(y)$ integers for the list, and so the set of integers of the form $x00y$ generate an additional $9 \times T(y)$ integers since there are 9 choices for x. When we add $T(y)$ for all choices of y, we get S_1, so the integers $x00y$ contribute $9 \times S_1$ additional integers to the list.

By similar reasoning, integers of the form $x0ya$ contribute $9 \times S_2$ extra integers, and integers of the form $xy0z$ contribute an extra $9^2 \times S_1$ integers. Thus, $S_4 = 9000 + 9(S_1 + S_2) + 9^2(S1) = 10620$.

Consider S_n where $n \geq 4$. There are $9 \times 10^{n-1}$ n-digit integers in the list before the 0s are erased. Integers that generate at least one more integer when 0s are erased consist of d non-zero digits (with $1 \leq d \leq n-2$) followed by r zeroes (with $1 \leq r \leq n-d-1$), followed by an integer with $n - d - r$ digits. We can write such an integer as

$$N = x_1 x_2 \cdots x_d 0 \cdots 0 y a_2 a_3 \cdots a_{n-d-r}$$

where $x_1, x_2, \ldots, x_d, y$ are non-zero digits. Note that

$$T(N) = T(x_1 x_2 \cdots x_d 0 \cdots 0 y a_2 a_3 \cdots a_{n-d-r}) = 1 + T(y a_2 a_3 \cdots a_{n-d-r})$$

There are 9^d combinations of digits $x_1 x_2 \cdots x_d$ that go before the r zeroes, since there are d non-zero digits. Given a specific combination of these d digits, the number of new integers generated for the list across all possible $n - d - r$ final digits is S_{n-d-r} because the digits $y a_2 a_3 \cdots a_{n-d-r}$ range through all possible $n - d - r$ positive integers. In other words, given a specific n, d and r, the number of additional integers generated is

$$9^d S_{n-d-r}$$

since there are 9^d ways the integer can start and, for each of these, S_{n-d-r} ways in which the last $n - d - r$ digits generate new integers. For example, if $n = 5$ and $d = 2$, then we can have $r = 1$ or $r = 2$.

Therefore,

$$\begin{aligned} S_5 &= 9 \times 10^4 + 9(S_1 + S_2 + S_3) + 9^2(S_1 + S_2) + 9^3(S_1) \\ &= 90\,000 + 9(9 + 90 + 981) + 9^2(9 + 90) + 9^3(9) \\ &= 114\,300 \end{aligned}$$

$$\begin{aligned} S_6 &= 9 \times 10^5 + 9(S_1 + S_2 + S_3 + S_4) + 9^2(S_1 + S_2 + S_3) + 9^3(S_1 + S_2) + 9^4(S_1) \\ &= 900\,000 + 9(9 + 90 + 981 + 10\,620) + 9^2(9 + 90 + 981) + 9^3(9 + 90) + 9^4(9) \\ &= 1\,224\,000 \end{aligned}$$

$$\begin{aligned} S_7 &= 9 \times 10^6 + 9(S_1 + S_2 + S_3 + S_4 + S_5) + 9^2(S_1 + S_2 + S_3 + S_4) \\ &\quad + 9^3(S_1 + S_2 + S_3) + 9^4(S_1 + S_2) + 9^5(S_1) \\ &= 9\,000\,000 + 9(9 + 90 + 981 + 10\,620 + 114\,300) + 9^2(9 + 90 + 981 + 10\,620) \\ &\quad + 9^3(9 + 90 + 981) + 9^4(9 + 90) + 9^5(9) \\ &= 13\,050\,000 \end{aligned}$$

$$\begin{aligned} S_8 &= 9 \times 10^7 + 9(S_1 + S_2 + S_3 + S_4 + S_5 + S_6) \\ &\quad + 9^2(S_1 + S_2 + S_3 + S_4 + S_5) + 9^3(S_1 + S_2 + S_3 + S_4) \\ &\quad + 9^4(S_1 + S_2 + S_3) + 9^5(S_1 + S_2) + 9^6(S_1) \\ &= 90\,000\,000 + 9(9 + 90 + 981 + 10\,620 + 114\,300 + 1\,224\,000) \\ &\quad + 9^2(9 + 90 + 981 + 10\,620 + 114\,300) + 9^3(9 + 90 + 981 + 10\,620) \\ &\quad + 9^4(9 + 90 + 981) + 9^5(9 + 90) + 9^6(9) \\ &= 138\,600\,000 \end{aligned}$$

ANSWER: 138 600 000

Part B

1. (a) Substituting in the coordinates of the point into the equation gives $-9 = 6p + p^2$. Rearranging this equation gives $0 = p^2 + 6p + 9 = (p + 3)^2$. Therefore, $p = -3$.

   (b) Substituting each of the three points into the equation gives the following three equations:

$$\begin{aligned} 5 &= a - b + c \\ 4 &= c \\ 11 &= a + b + c. \end{aligned}$$

   Substituting $4 = c$ into the other two equations and rearranging a little gives

$$\begin{aligned} 1 &= a - b \\ 7 &= a + b. \end{aligned}$$

   Adding these two equations together gives $2a = 8$ and therefore $a = 4$. Substituting $a = 4$ into the last equation yields $b = 3$.

   Therefore, $a = 4$, $b = 3$, $c = 4$.

   (c) *Solution 1*

   Substituting the coordinates of both points into the equation of the parabola gives the two equations

$$\begin{aligned} 5 &= (d - 1)^2 + q \\ 5 &= (d + 5)^2 + q \end{aligned}$$

   Subtracting the two questions gives

$$\begin{aligned} 0 &= (d - 1)^2 + q - [(d + 5)^2 + q] \\ &= d^2 - 2d + 1 + q - d^2 - 10d - 25 - q \\ &= -12d - 24 \end{aligned}$$

   and so $12d = -24$, or $d = -2$. The parabola passes through $(-2, 5)$, so we have

$$5 = (-2 - 1)^2 + q$$

   which can be solved for q to get $q = -4$.

   Solution 2

   Since the points $(d, 5)$ and $(d + 6, 5)$ have the same y-coordinate, they must be mirror images of each other in the axis of symmetry. Therefore, the axis of symmetry is $x = d+3$ since $d + 3$ is the average of the two x coordinates.

   The axis of symmetry contains the vertex of the parabola, so the x-coordinate of the vertex of the parabola must be $x = d + 3$.

   The equation of the parabola is given in vertex form, so we can immediately get from the equation $y = (x-1)^2+q$ that the x-coordinate of the vertex is $x = 1$. Therefore, $d+3 = 1$ and $d = -2$. The parabola passes through $(-2, 5)$, so we have

$$5 = (-2 - 1)^2 + q$$

   which can be solved for q to get $q = -4$.

2. (a) Since AD is parallel to BC, $\angle BCA = \theta$. By the Pythagorean theorem, $AC^2 = 2^2 + 3^2$ and so $AC = \sqrt{13}$. Therefore, $\sin\theta = \dfrac{AB}{AC} = \dfrac{3}{\sqrt{13}}$.

(b) *Solution 1*

Let x be the length of TR and let y be the length of PR. Let $\alpha = \angle PTR$. The sine law applied to $\triangle PTR$ gives

$$\frac{\sin 45^\circ}{x} = \frac{\sin\alpha}{y}. \tag{1}$$

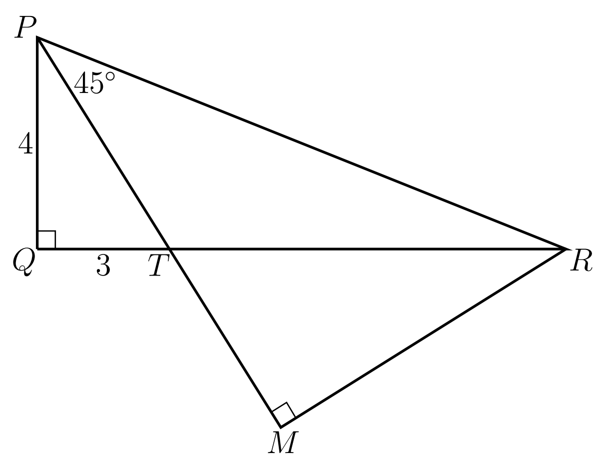
By the Pythagorean theorem, we get $y = \sqrt{4^2 + (3+x)^2} = \sqrt{25 + 6x + x^2}$ and also $PT = \sqrt{3^2 + 4^2} = 5$. By right-angle trigonometry and that $\angle PTQ = 180^\circ - \alpha$, we get $\sin\alpha = \sin(180^\circ - \alpha) = \dfrac{4}{5}$. Recall that $\sin 45^\circ = \dfrac{1}{\sqrt{2}}$. Substituting all of this into Equation (1) and rearranging gives

$$\begin{aligned}
\frac{4}{5\sqrt{25+6x+x^2}} &= \frac{1}{\sqrt{2}x} \\
4\sqrt{2}x &= 5\sqrt{25+6x+x^2} \\
32x^2 &= 25(25+6x+x^2) \\
7x^2 - 150x - 625 &= 0.
\end{aligned}$$

The roots of this quadratic are $x = \dfrac{-25}{7}$ and $x = 25$. Since x is a length, we reject the negative solution and conclude that $TR = 25$.

Solution 2

Extend PT to a point M so that $\angle PMR = 90^\circ$.



By the Pythagorean theorem, $PT = \sqrt{4^2 + 3^2} = 5$.

In $\triangle PMR$, there is a right angle and a 45° angle, so $\triangle PMR$ must be a right-isosceles triangle, and so $RM = PM$. If we let $TM = z$, then we have $PM = RM = 5 + z$.

In $\triangle MRT$ and $\triangle QTP$, we have $\angle MTR = \angle QTP$ by opposite angles. These triangles also both have a right angle, so we conclude that $\triangle MRT$ and $\triangle QTP$ are similar. Thus, $\dfrac{RM}{TM} = \dfrac{PQ}{TQ}$, and so $\dfrac{5+z}{z} = \dfrac{4}{3}$. Rearranging this equation gives $15 + 3z = 4z$, and so $z = 15$.

Again using similar triangles, $\dfrac{TR}{TM} = \dfrac{TP}{TQ}$, so $TR = \dfrac{TM \cdot PT}{QT} = \dfrac{15 \cdot 5}{3} = 25$.

Solution 3

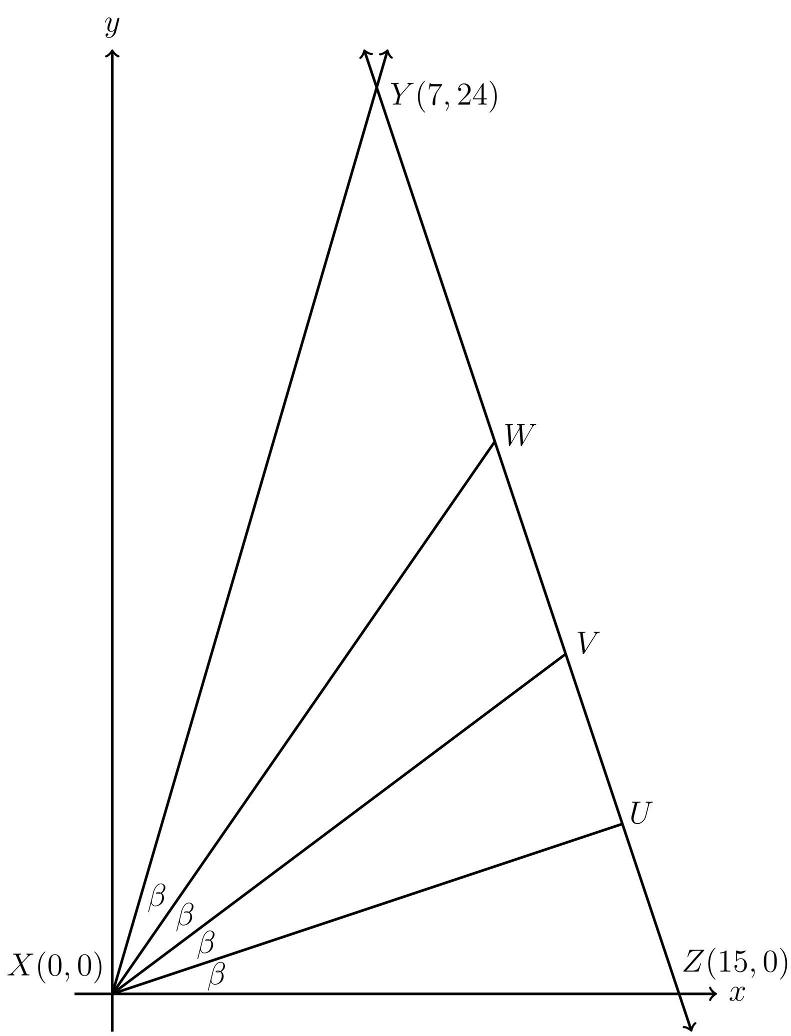
If we set $x = TR$, then $\tan\angle QPR = \dfrac{3+x}{4}$. As well, $\tan\angle QPR = \tan(\angle QPT + \angle TPR)$. From the diagram, $\tan\angle QPT = \dfrac{3}{4}$, and $\tan\angle TPR = \tan 45^\circ = 1$. Using the formula $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$, we get

$$\begin{aligned}\frac{3+x}{4} &= \tan\angle QPR \\ &= \tan(\angle QPT + \angle TPR) \\ &= \frac{\tan\angle QPT + \tan\angle TPR}{1 - \tan\angle QPT \tan\angle TPR} \\ &= \frac{\frac{3}{4}+1}{1-\frac{3}{4}\cdot 1} \\ &= 7\end{aligned}$$

and so $\dfrac{3+x}{4} = 7$, which can be solved for x to get $x = 25$.

(c) *Solution 1*

Let U and V be on YZ so that XU and XV *trisect* $\angle WXZ$. Specifically, along YZ, W is closest to Y, V is next closet to Y, and U is farthest from Y. This will cause $\angle WXV = \angle VXU = \angle UXZ = \beta$. Then $\angle WXZ = 3\beta$, so $\angle WXY = \beta$ as well.



In general, the slope of a line through the origin is equal to tan of the angle made by the line and the x-axis. Thus, the line WX has slope $\tan 3\beta$. If we can find $\tan 3\beta$, then we can find the equation of WX, then find the coordinates of W by intersecting with YZ.

Towards the goal of computing $\tan 3\beta$, observe that $\tan(\angle YXZ) = \tan 4\beta = \dfrac{24}{7}$. Using the double-angle formula for the tangent function, we have that

$$\tan 4\beta = \frac{2\tan 2\beta}{1 - \tan^2 2\beta}$$

and if we let $\tan 2\beta = m$, we have $\dfrac{24}{7} = \dfrac{2m}{1-m^2}$.

Cross multiplying gives $24(1-m^2) = 7(mx)$, which can be rearranged to get the quadratic $24m^2 + 14m - 24 = 0$. By the quadratic formula, the roots of this quadratic are $\dfrac{-7 \pm 25}{24}$.

Since 2β is an angle in the first quadrant, $\tan 2\beta$ is positive, so $\tan 2\beta = \dfrac{-7+25}{24} = \dfrac{3}{4}$.

We can repeat the same process with $\tan 2\beta = \dfrac{2\tan\beta}{1-\tan^2\beta}$, this time setting $\tan\beta = n$ to get $\dfrac{3}{4} = \dfrac{2n}{1-n^2}$. This is equivalent to the quadratic $3n^2 + 8n - 3 = 0$. Using the quadratic formula, its only positive root is $n = \dfrac{-8+10}{6} = \dfrac{1}{3}$.

We can now use the sum formula for the tan function to get

$$\begin{aligned}\tan 3\beta &= \frac{\tan\beta + \tan 2\beta}{1 - \tan\beta\tan 2\beta} \\ &= \frac{\frac{1}{3} + \frac{3}{4}}{1 - \frac{1}{3}\cdot\frac{3}{4}} \\ &= \frac{13}{9}\end{aligned}$$

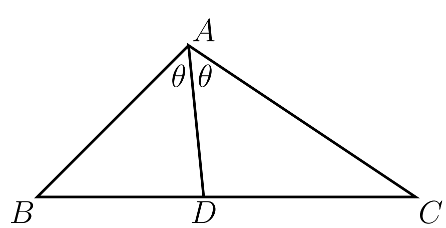
Therefore, line segment WX is part of the line with equation $y = \dfrac{13}{9}x$. One can also check that line segment YZ is part of the line with equation $y = -3x + 45$. As mentioned earlier, W is the point of intersection of these two lines.

The x-coordinate of W is the solution to $-3x + 45 = \dfrac{13}{9}x$, which is $\dfrac{81}{8}$. The y-coordinate is $y = \dfrac{13}{9} \cdot \dfrac{81}{8} = \dfrac{117}{8}$.

The coordinates of W are $\left(\dfrac{81}{8}, \dfrac{117}{8}\right)$.

Solution 2

Using the same diagram as in Solution 1, we will instead approach this problem using the *Angle Bisector Theorem*. That is, if in $\triangle ABC$, point D is on BC so that AD bisects $\angle BAC$, then $\dfrac{AB}{AC} = \dfrac{DB}{DC}$.



In $\triangle XYZ$, the Angle Bisector Theorem implies $\dfrac{XY}{XZ} = \dfrac{VY}{VZ}$. By the Pythagorean theorem, $XY = \sqrt{24^2 + 7^2} = 25$. It is given that, $XZ = 15$, so we have $\dfrac{25}{15} = \dfrac{VY}{VZ}$ which is equivalent to $VZ = \dfrac{3}{5}VY$.

By the Distance formula,

$$YZ = \sqrt{(7-15)^2 + (24-0)^2} = \sqrt{64+576} = \sqrt{640} = 8\sqrt{10}$$

Since $YZ = VY + VZ$, we can substitute $VZ = \dfrac{3}{5}VY$ to get

$$8\sqrt{10} = VY + \frac{3}{5}VY$$

and solve for VY to get $VY = 5\sqrt{10}$. It follows that $VZ = 3\sqrt{10}$.

We can also find the coordinates of V. Since V is $\dfrac{5}{8}$ of the way from Y to Z along YZ, the x-coordinate of V is $\dfrac{5}{8}$ of the way from the x-coordinate of Y (which is 7) to the x-coordinate of Z (which is 15). Therefore, the x-coordinate of V is $7 + \dfrac{5}{8}(15-7) = 12$. By similar reasoning, the y-coordinate of V is $24 + \dfrac{5}{8}(0-24) = 9$.

The coordinates of V are $(12, 9)$, so the length of XV is $\sqrt{12^2 + 9^2} = 15$.

Using the Angle Bisector theorem on $\triangle XYV$, we now have that $\dfrac{XV}{XY} = \dfrac{WV}{WY}$, and so $\dfrac{15}{25} = \dfrac{WV}{WY}$ or $WV = \dfrac{3}{5}WY$. Using that $WY + WV = YV = 5\sqrt{10}$ and substituting, gives

$$5\sqrt{10} = WY + \frac{3}{5}WY$$

which can be solved to get $WY = \dfrac{25\sqrt{10}}{8}$.

Now we have $\dfrac{YW}{YZ} = \dfrac{\frac{25\sqrt{10}}{8}}{8\sqrt{10}} = \dfrac{25}{64}$, so by similar reasoning to that which was used to find the coordinates of V, the x-coordinate of W is $7 + \dfrac{25}{64}(15-7) = \dfrac{81}{8}$ and the y-coordinate of W is $24 + \dfrac{25}{64}(0-24) = \dfrac{117}{8}$.

The coordinates of W are $\left(\dfrac{81}{8}, \dfrac{117}{8}\right)$.

3. (a) From the conditions given in the setup of the problem, we have

$$\begin{aligned} c_1 &= 1 \\ c_2 &= b_1c_1 \\ c_3 &= b_1c_2 + b_2c_1 \\ c_4 &= b_1c_3 + b_2c_2 + b_3c_1 \end{aligned}$$

we also have that $b_1 = 1$, $b_2 = 2$, and $b_3 = 4$. Substituting $b_1 = 1$ and $c_1 = 1$ into $c_2 = b_1c_1$ gives $c_2 = (1)(1) = 1$. Now we have

$$c_3 = b_1c_2 + b_2c_1 = (1)(1) + (2)(1) = 3$$

and finally

$$c_4 = b_1c_3 + b_2c_2 + b_3c_1 = (1)(3) + (2)(1) + (4)(1) = 9$$

so $c_2 = 1$, $c_3 = 3$, and $c_4 = 9$.

(b) *Solution 1*

Suppose $b_1 = a$ and the common ratio is r, so that $b_n = ar^{n-1}$ for all $n \geq 1$. The first few c_i can be computed using the relationship between a sequence and its complement.

$$\begin{aligned}
c_1 &= 1\\
c_2 &= b_1c_1\\
&= (a)(1)\\
&= a\\
c_3 &= b_1c_2 + b_2c_1\\
&= (a)(a) + (ar)(1)\\
&= a(a+r)\\
c_4 &= b_1c_3 + b_2c_2 + b_3c_1\\
&= (a)\big(a(a+r)\big) + (ar)(a) + \big(ar^2\big)(1)\\
&= a^2(a+r) + a^2r + ar^2\\
&= a(a^2 + ar + ar + r^2)\\
&= a(a+r)^2
\end{aligned}$$

From $c_2 = a$, $c_3 = a(a+r)$, and $c_4 = a(a+r)^2$, we have $c_3 = (a+r)c_2$ and $c_4 = (a+r)c_3$, so we conjecture that the constant we seek is $t = a + r$.

We will proceed to prove that $c_n = (a+r)c_{n-1}$ for all $n \geq 3$ by induction. We have already shown this for $n = 3$ and $n = 4$. Assume that, for some k, we have $c_3 = (a+r)c_2$, $c_4 = (a+r)c_3$, $c_5 = (a+r)a_4$, and so on up to $c_k = (a+r)c_{k-1}$. Using this assumption, we will deduce that $c_{k+1} = (a+r)c_k$.

By the defining identity of c_{k+1} as well as our assumption above, we have

$$\begin{aligned}
c_{k+1} &= b_1c_k + b_2c_{k-1} + \cdots + b_{k-2}c_3 + b_{k-1}c_2 + b_kc_1\\
&= b_1(a+r)c_{k-1} + b_2(a+r)c_{k-2} + \cdots + b_{k-2}(a+r)c_2 + b_{k-1}c_2 + b_kc_1\\
&= (a+r)\big(b_1c_{k-1} + b_2c_{k-2} + \cdots + b_{k-2}c_2\big) + b_{k-1}c_2 + b_kc_1 \qquad (*)
\end{aligned}$$

Again using the definition from the question, we have

$$\begin{aligned}
c_k &= b_1c_{k-1} + b_2c_{k-2} + \cdots + b_{k-2}c_2 + b_{k-1}c_1\\
c_k - b_{k-1}c_1 &= b_1c_{k-1} + b_2c_{k-2} + \cdots + b_{k-2}c_2
\end{aligned}$$

The right side of this expression appears in $(*)$, so we can substitute to get

$$\begin{aligned}
c_{k+1} &= (a+r)\big(b_1c_{k-1} + b_2c_{k-2} + \cdots + b_{k-2}c_2\big) + b_{k-1}c_2 + b_kc_1\\
&= (a+r)\big(c_k - b_{k-1}c_1\big) + b_{k-1}c_2 + b_kc_1\\
&= (a+r)c_k - (a+r)b_{k-1}c_1 + b_{k-1}c_2 + b_kc_1 \qquad (**)
\end{aligned}$$

Finally, we can use known expressions for b_i and c_i to show that

$$\begin{aligned}
-(a+r)b_{k-1}c_1 + b_{k-1}c_2 + b_kc_1 &= -(a+r)ar^{k-2} + ar^{k-2}a + ar^{k-1}\\
&= -a^2r^{k-2} - ar^{k-1} + a^2r^{k-2} + ar^{k-1}\\
&= 0
\end{aligned}$$

Substituting this into $(**)$, we get $c_{k+1} = (a+r)c_k$. This completes the induction, so we have shown that $c_n = (a+r)c_{n-1}$ for all $n \geq 3$.

Solution 2

Given $n \geq 3$, we have the following two equations. Note that $n \geq 3$ is required for $n-1 \geq 2$, which is what allows us to write down the second of these two equations.

$$\begin{aligned} c_n &= b_1c_{n-1} + b_2c_{n-2} + b_3c_{n-3} + \cdots + b_{n-2}c_2 + b_{n-1}c_1 \\ c_{n-1} &= b_1c_{n-2} + b_2c_{n-3} + \cdots + b_{n-3}c_2 + b_{n-2}c_1 \end{aligned}$$

It is given that $b_1, b_2, b_3, \ldots$ is a geometric sequence. If we let r be its common ratio, then we have $rb_k = b_{k+1}$ for each $k \geq 1$. Thus, we can multiply both sides of the second equation above through by r to get

$$\begin{aligned} rc_{n-1} &= (rb_1)c_{n-2} + (rb_2)c_{n-3} + \cdots + (rb_{n-3})c_2 + (rb_{n-2})c_1 \\ &= b_2c_{n-2} + b_3c_{n-3} + \cdots + b_{n-2}c_2 + b_{n-1}c_1 \end{aligned}$$

and so we get the following two equations:

$$\begin{aligned} c_n &= b_1c_{n-1} + b_2c_{n-2} + b_3c_{n-3} + \cdots + b_{n-2}c_2 + b_{n-1}c_1 \\ rc_{n-1} &= \phantom{b_1c_{n-1} + {}} b_2c_{n-2} + b_3c_{n-3} + \cdots + b_{n-2}c_2 + b_{n-1}c_1 \end{aligned}$$

The right sides of these two equations are almost identical. Indeed, subtracting the second from the first gives $c_n - rc_{n-1} = b_1c_{n-1}$. Since $b_1 = ar^{1-1} = a$, this can be rearranged to get $c_n = (a+r)c_{n-1}$.

We have shown that $c_n = (a+r)c_{n-1}$ for all $n \geq 3$. Thus, the required identity is satisfied with $t = a + r$ where a and r are the constants such that $b_n = ar^{n-1}$ for all $n \geq 1$.

(c) Observe that the sequence $b_1, b_2, b_3, \ldots$ being arithmetic means $b_{m+2} - b_{m+1} = b_{m+1} - b_m$ for every $m \geq 1$, which can be rearranged to get $b_{m+2} - 2b_{m+1} + b_m = 0$ for all $m \geq 1$.

For $n \geq 4$, we have

$$\begin{aligned} c_n &= b_1c_{n-1} + b_2c_{n-2} + b_3c_{n-3} + b_4c_{n-4} + \cdots + b_{n-2}c_2 + b_{n-1}c_1 && (1) \\ c_{n-1} &= \phantom{b_1c_{n-1} + {}} b_1c_{n-2} + b_2c_{n-3} + b_3c_{n-4} + \cdots + b_{n-3}c_2 + b_{n-2}c_1 && (2) \\ c_{n-2} &= \phantom{b_1c_{n-1} + b_2c_{n-2} + {}} b_1c_{n-3} + b_2c_{n-4} + \cdots + b_{n-4}c_2 + b_{n-3}c_1 && (3) \end{aligned}$$

where the equations above are written so that the terms involving c_i are vertically aligned for each i from 1 through $n-3$.

Consider what happens if we compute Equation (1) minus twice Equation (2) plus Equation (3). On the right side of the resulting equation, the terms involving c_1 are

$$b_{n-1}c_1 - 2b_{n-2}c_1 + b_{n-3}c_1 = c_1\big(b_{n-1} - 2b_{n-2} + b_{n-3}\big) = c_1(0) = 0$$

by the remark above. More generally, if k is between 1 and $n-3$ inclusive, then

$$b_{n-k}c_k - 2b_{n-k-1}c_k + b_{n-k-2}c_k = c_k(b_{n-k} - 2b_{n-k-1} + b_{n-k-2}) = c_k(0)$$

Note that $k \leq n-3$, so $n-k-2 \geq 1$, so we can indeed use this identity for all such k.

Thus, if we compute Equation (1) minus twice Equation (2) plus Equation (3), all terms containing c_i for i from 1 through $n-3$ cancel. After this cancellation, we are left with the equation

$$c_n - 2c_{n-1} + c_{n-2} = b_1 c_{n-1} + b_2 c_{n-2} - 2b_1 c_{n-2} \tag{4}$$

If $n = 2028$, then using the information given in the problem statement, we have $c_n = 3$, $c_{n-1} = -3$, $c_{n-2} = 0$, so Equation (4) gives

$$3 - 2(-3) + 0 = b_1(-3) + b_2(0) - 2b_1(0)$$

which can be simplified to get $9 = -3b_1$ or $b_1 = -3$.

If $n = 2027$, then $c_n = -3$, $c_{n-1} = 0$, and $c_{n-2} = 3$, so Equation (4) gives

$$-3 - 2(0) + 3 = b_1(0) + b_2(3) - 2b_1(3)$$

or $0 = 3b_2 - 6b_1$. We just showed that $b_1 = -3$, so we can substitute to get $3b_2 = -18$ or $b_2 = -6$.

We are given that $b_1, b_2, b_3, \ldots$ is an arithmetic sequence, and we now know that $b_1 = -3$ and $b_2 = -6$. It follows that the common difference of the sequence is $-6 - (-3) = -3$, so $b_n = -3 - 3(n-1)$ for all $n \geq 1$. Therefore, $b_{2025} = -3 - 3(2024) = -6075$.