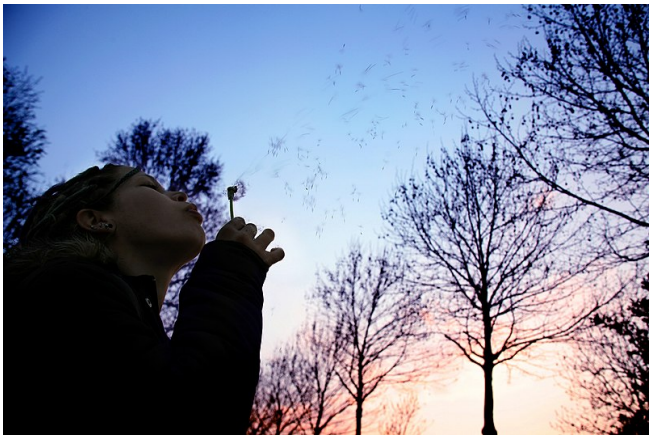
Problems with Purpose

Volume 2.01




Source: Top left photo: https://commons.wikimedia.org/wiki/File:Woman_blowing_dandelion_seeds_in_Paris.jpg

Source: Top right photo: https://commons.wikimedia.org/wiki/File:Sierra%27s_birthday,_campfire_(16209461406).jpg

Source: Bottom left photo: https://commons.wikimedia.org/wiki/File:As08-16-2593.jpg

Source: Bottom right photo by Judith Koeller

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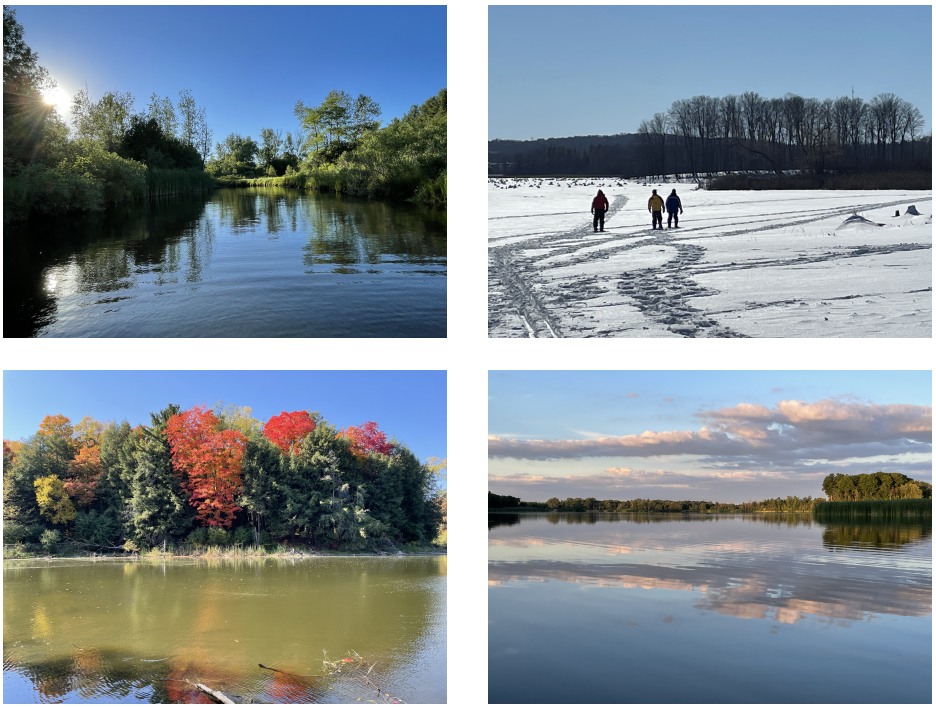
Introduction

The Centre for Education in Mathematics and Computing is situated on the traditional territory of the Neutral, Anishinaabeg and Haudenosaunee peoples. As part of the University of Waterloo campus, the CEMC is located on the Haldimand Tract, the land promised and given to the Six Nations, that includes six miles on either side of the Grand River. We seek to work in the spirit of the University of Waterloo's Indigenous Strategic Plan and the Faculty of Math's Equity and Inclusive Communities Principles. We partner with educators and work with students who are located across Turtle Island and around the world.



Source: Photos in the Grand River watershed by Judith Koeller

Over the last few years, the Centre for Education in Mathematics and Computing has had many conversations with teachers who are Indigenous, and teachers who teach in Indigenous communities. We talked with folks both from Six Nations and across Turtle Island about how the CEMC could support the calls of the Truth and Reconciliation Commission.

A recurring theme was the importance of exploring mathematics through hands-on activities, with concrete connections to the world around us, experiencing mathematics as a tool that helps communities and the world. In fact, these themes resonate for many students of mathematics, whether or not they are Indigenous.

Problems With Purpose is a collection of mathematical problems to be used in grades 4 to 12. Each problem is meant to highlight the connection between mathematics and our collective responsibility to present and future generations. Topics include protecting water, trees and animals, fostering the Cree language, demographic trends, and reducing poverty. We welcome your feedback at cemc@uwaterloo.ca.

Acknowledgements

We are grateful to the many folks whose insights, ideas and skills contributed to this project:

Advisory:	Jean Becker	Kathleen Couch	Edward Doolittle	Nathan Rowbottom
	Mark Skanks	Rachel White		
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	Ryan McGrath	Jen Nelson	Lata Punetha	Nicholas Rollick
	Ian VanderBurgh	Christine Vender		

Water

Source: https://commons.wikimedia.org/wiki/File:Ocean_water_on_sand.jpg

Drinking Water

According to Statistics Canada, the average amount of drinkable (often called "potable") water used per person was 485 L per day in 2011.

1. The average dropped to 466 L per day in 2013. By what percent did the average decline from 2011 to 2013?

   Source: https://www.statcan.gc.ca/en/dai/smr08/2017/smr08_215_2017

2. In 2013, a typical charge for tap water in a Canadian city was $2 for 1000 L. If a person used an average of 466 L of tap water per day at this price, how much would they spend on water for the whole year?

3. Some folks in remote communities do not have access to drinkable tap water. Suppose that a store charged $5.49 for 15 L of bottled water in 2013. How much would a person spend on water per day in 2013 if they bought 466 L of water at this price?
   How much more per year would a person spend on drinkable water if they bought it from a store vs. paying for city tap water?

4. Search for the latest Statistics Canada data on drinkable water use. How has the average changed since 2013?

Source: https://commons.wikimedia.org/wiki/File:Water_splashing_out_of_a_full_clear,_glass_cup._(15055172195).jpg

Solution to Drinking Water

1. Since $\frac{466}{485} \times 100\% \approx 96\%$, and $100\% - 96\% = 4\%$, from 2011 to 2013, the average amount of drinking water used per person declined by about 4%.

2. The cost of tap water was $\frac{\$2}{1000L} = \$0.002$ per L. Since there were 365 days in 2013, the total amount of tap water used in the year was 365×466 L $= 170\,090$ L.
   The total cost of the tap water was $170\,090$ L $\times \$0.002/L = \$340.18$.

3. The cost of bottled water was $\frac{\$5.49}{15L} = \$0.366$ per L. As in the previous part, the total amount of water used in the year was 365×466 L $= 170\,090$ L.
   The total cost of the bottled water was $170\,090$ L $\times \$0.366/L = \$62\,252.94$.
   The water from the store would have cost $\$62\,252.94 - \$340.18 = \$61\,912.76$ more per year.

4. As of February 2024, Statistics Canada lists the average amount in 2021 as 401 L per day.

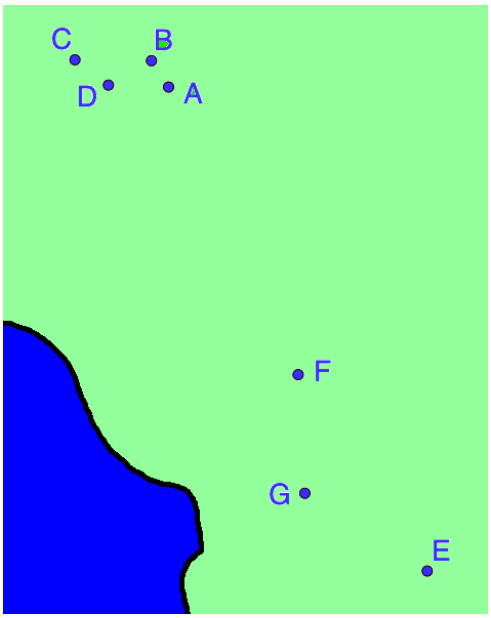
Earth

Source: https://commons.wikimedia.org/wiki/File:Grib_skov.jpg

Bear Sightings

Goal: Given a set of locations where a bear was recently seen, try to locate the bear.

1. On the map to the right, shade a small circle (the size of a coin) that you think is "close" to all of the points A, B, C, D, E, F, and G.
2. Imagine you had to teach a computer how to choose a region close to those points. How would you describe what to do? Did you add any lines to the map, or compute the distance between pairs of points, or use averages?



3. Suppose that a bear has been sighted at each of the points and a scientist is trying to locate the bear. Where on the map should the scientist focus their search?

Source: https://commons.wikimedia.org/wiki/File:Kamchatka_Brown_Bear_near_Dvuhyurtochnoe_on_2015-07-23.jpg

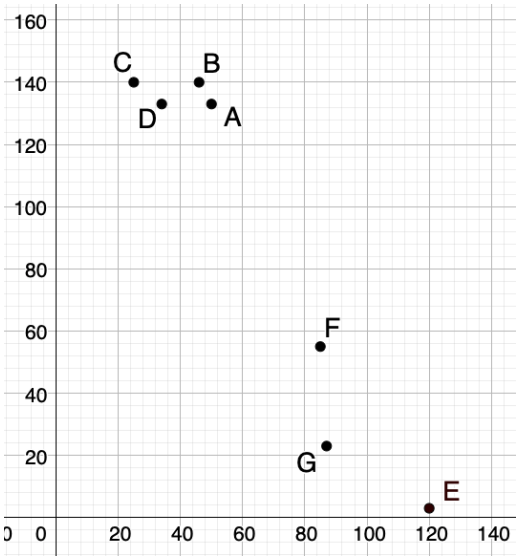
4. The **centroid** of set of points is a point M, whose x coordinate in the average of the other points' x coordinates, and whose y coordinates is the average of the other points' y coordinates.

   Given the coordinates of A, B, C, D, E, F, and G, calculate the coordinates of their centroid, M.

   Plot the centroid, M, on the map to the right.

   Is the centroid within the region you shaded in #1?

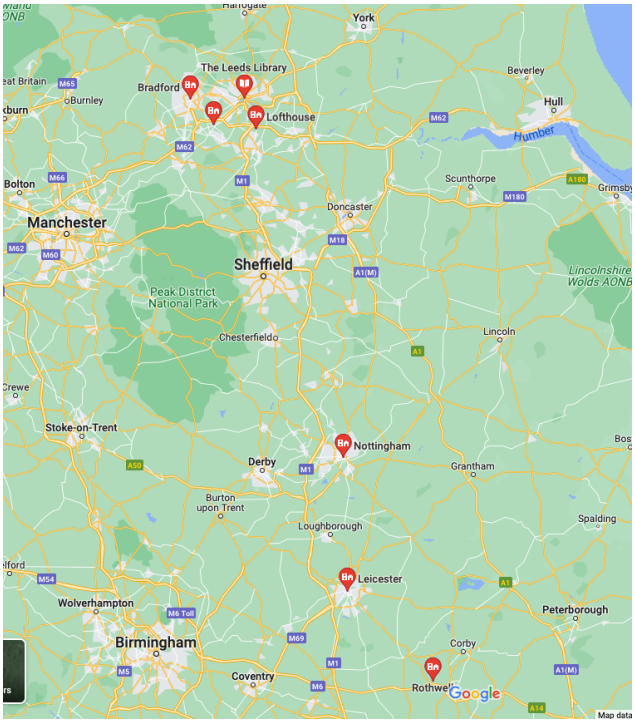
Point	x	y
A	50	133
B	46	140
C	25	140
D	34	133
E	120	3
F	85	55
G	87	23
Total		
M		



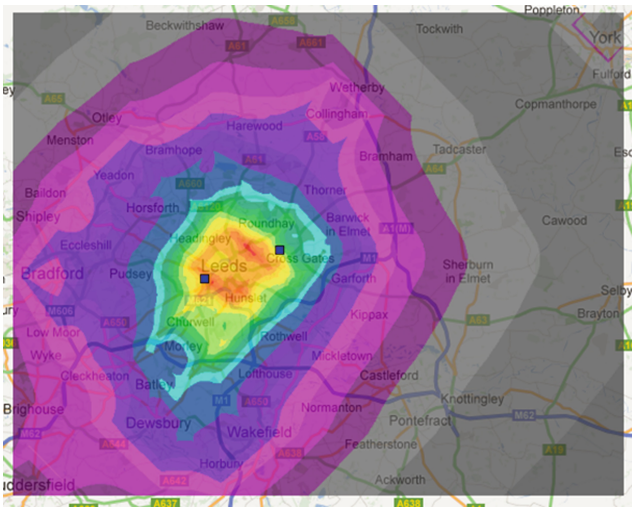
A related mathematical technique: Geoprofiling

Many police forces study the locations of past incidents to try to predict where future crime may occur. You can often view the maps online.

The points on the bear sightings map are arranged in a way similar to places in the UK where a series of related crimes were committed.



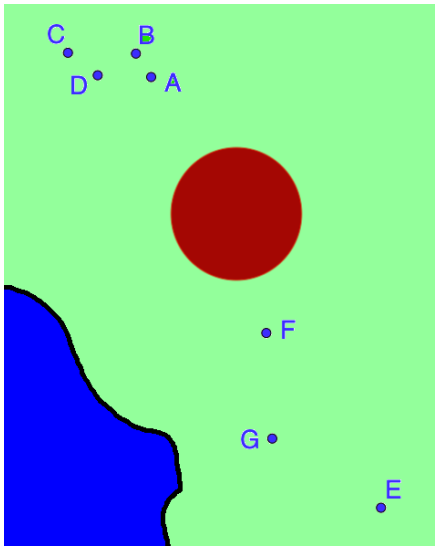
To narrow down their search for the criminal, police used a **geoprofiling algorithm** to identify the most likely place where the criminal lived. Their algorithm generated a map like the following. The criminal was found within the "hot spot" of the map. What do you think the colours on the map mean?



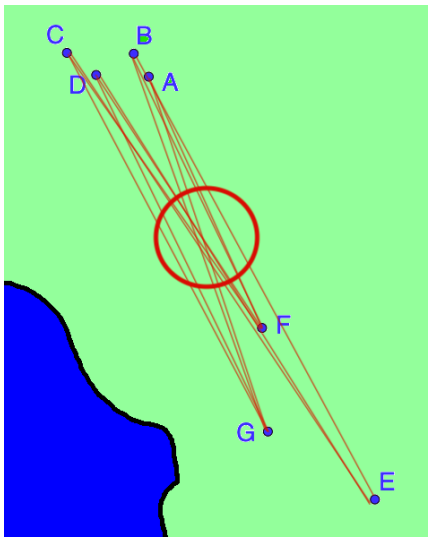
Read more about geoprofiling in this paper. There are social justice concerns around geoprofiling. Read why many mathematicians are opposed to this practice.

Solution to Bear Sightings

1. The circle shown is close to all points A, B, C, D, E, F, and G.



2. Since points A, B, C, D form a group and points E, F, G form a group, one option is to draw lines between the groups of points. We could focus on the region indicated in the circle, where many of the lines cross, as shown.

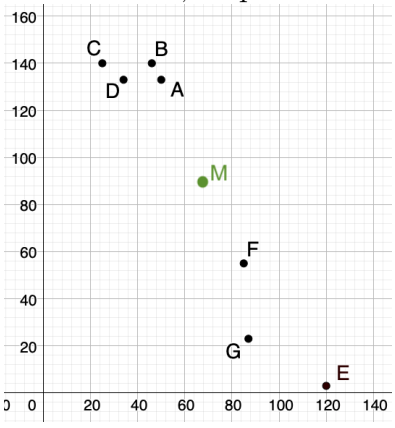


3. The regions we identified in #1 and #2 would be a good place to look for the bear, since it may be passing among all the different points where it was sighted.

4. The sum of the x coordinates is $50+46+25+34+120+85+87 = 447$. Thus, their average is $447 \div 7 \approx 63.9$.
The sum of the y coordinates is $133+140+140+133+3+55+23 = 627$. Thus, their average is $627 \div 7 \approx 89.6$.

Point	x	y
A	50	133
B	46	140
C	25	140
D	34	133
E	120	3
F	85	55
G	87	23
Total	447	627
M	63.9	89.6

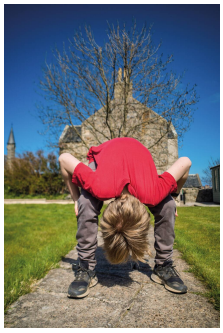
A good place to look for the bear is at M, as plotted below.



Notice that the point $M \approx (63.9, 89.6)$ is in the "hot spots" that we drew in #1 and #2. Calculating the coordinates of M like we did helps the computer to find a point that is close to A, B, C, D, E, F, and G.

Tree Height

When managing forests, it can be helpful to know the height of a tree, but it isn't easy to measure tree height directly. The following method uses geometry/trigonometry to estimate tree height.



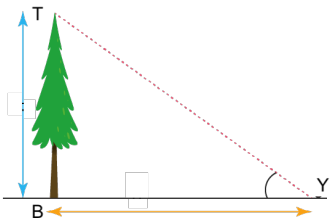
Step 1: Walk away from the tree, but at regular intervals bend down and look through your legs back up towards the tree.

Step 2: Continue doing this until you can just see the top of the tree.

Step 3: Measure the distance along the ground from your back to the tree.

Step 4: The distance you have measured is roughly equal to the tree's height.

1. In the diagram, the sides of the triangle are the ground (YB), the tree (BT), and your line of sight when looking at the top of the tree (YT).
   Which two sides does the method assume are equal? In order for these two sides to be equal, what must $\angle BYT$ (your line of sight) equal?
   Source: https://www.cuemath.com/questions/how-to-measure-the-height-of-a-tree-using-trigonometry/



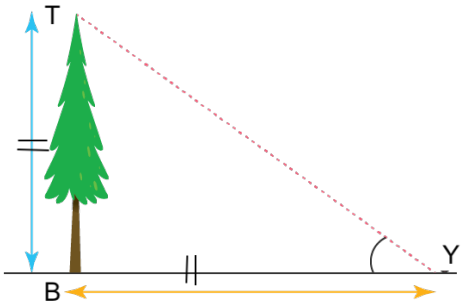
2. If you use this method, what is one way to ensure that your line of sight is as needed?

3. **Test the Method**.
   Put a piece of coloured tape as high up a building as you can reach to measure. Then, use the method to estimate the height of the tape. How close are your estimated height and the actual height?

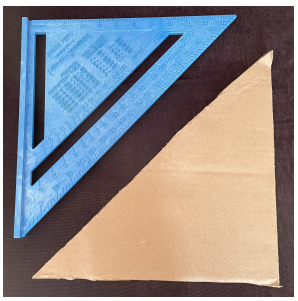
For more information, view
https://www.nts.org.uk/stories/can-you-estimate-the-height-of-a-tree.

Solution to Tree Height

1. This method assumes that $YB = BT$. For this to be true, we need $\angle BYT = \angle BTY$. Since the tree grows straight up from the ground, $\angle YBT = 90°$. Since the sum of the angles in the triangle equals 180°, it must be that $\angle BYT = 45°$. If you use this method to estimate the height of a tree, you should be careful to look up at an angle of 45° from the ground.



2. You could ensure that your line of sight is 45° by using a giant triangle with angles 45°, 45° and 90°. You can make such a triangle by cutting along the diagonal of a large cardboard square. Alternatively, use a carpenter's triangle. Place the triangle on the ground so its long side lines up with YT, then direct your line of sight along it.

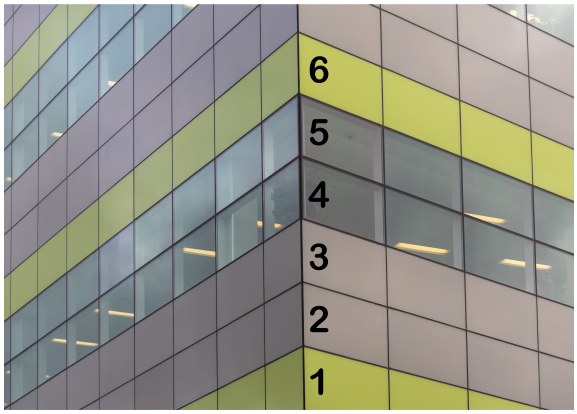


3. A common strategy in mathematics is to work backwards. In the top left of the photo, observe where we placed a piece of blue tape above the window frame, marked "T". The tape is 210 cm above the ground, so $BT = 210$ cm. To make $YB = BT$, we put another piece of tape on the ground at Y, at a distance of 210 cm from B. We tried to get both our eyes and a phone camera into a line of sight of 45° using the cardboard triangle, expecting to just see the top of the tape. We found this to be surprisingly difficult to do. While the mathematics of this method is correct, positioning yourself into a line of sight of 45° is harder then expected!

This leads to some interesting questions, such as under what conditions the wrong line of sight might make you over-estimate or under-estimate the height of a tree? If your line of sight is wrong, how far off is your estimate likely to be? Under what situations would this method be good enough, and how could it be improved with more specialized equipment?

Bird-Safe Glass

The building shown has some rows of panels in clear glass (like the fourth and fifth rows), some rows in green (like the first and sixth rows) and some rows in gray (like the second and third rows). The colours of the rows form a pattern that repeats after every five rows.



Suppose that the building needs to have all of its clear glass panels updated to bird-safe glass. Since birds often cannot see clear glass, bird-safe glass is made with a pattern on it that birds can see.

Source: Photo by Bangishimo

1. The building has 26 rows of panels in total. How many rows of clear glass does it have?
2. On each side of the building, each row contains 12 panels. How many panels in the whole building have clear glass?
3. The cost to replace one clear glass panel with bird-safe glass is $150. What is the total cost to replace all the building's clear glass panels with bird-safe glass?

About the Photographer

Bangishimo (They/Them) is an IndigiQueer Anishinaabe, originally from Couchiching First Nation located on Treaty #3 territory. Bangishimo's focus is creating space for communities to come together, allowing for Black, Indigenous and racialized voices to be heard. They are a co-founder of O:se Kenhionhata:tie, also known as Land Back Camp. Their advocacy and photography has allowed them to visit over sixteen countries; taking photos and sharing the stories of those they meet along the way. Bangishimo's work has been featured in numerous publications and displayed throughout Waterloo Region. Bangishimo won the Briarpatch Writing in the Margins Contest - Photography Category and was voted Best Photographer 2021 in the Community Edition.

Source: https://www.bangishimo.ca/

Solution to Bird-Safe Glass

1. Since the pattern of the rows repeats every 5 rows, then in 26 rows there will be 5 complete repeats of the pattern, with one extra row of green panels. Since we are counting only the rows of clear glass panels, we can ignore the extra row of green panels.

   In each repeat of the pattern there are 2 rows of clear glass panels. Therefore, in 5 repeats of the pattern there will be $2 \times 5 = 10$ rows of clear glass panels.

2. Since each row contains 12 panels, and there are 10 rows of panels, then on each side of the building there will be $12 \times 10 = 120$ clear glass panels. Since the building has 4 sides, there are $4 \times 120 = 480$ clear glass panels in total in the whole building.

3. Since each clear glass panel costs \$150 to replace, and there are 480 clear glass panels to replace, the total cost will be $\$150 \times 480 = \$72\,000$.

Here are some other ways you can protect birds so they don't fly into glass.

People

Source: https://commons.wikimedia.org/wiki/File:Sierra%27s_birthday,_campfire_(16209461406).jpg

Counting People by Gender

In the 2021 Statistics Canada census, people over the age of 15 reported their gender as:

Gender	Number of people
men	14,842,140
women	15,452,645
non-binary people	41,350

Source: https://www150.statcan.gc.ca/n1/daily-quotidien/220427/t002b-eng.htm

1. Determine the percentage of the population who reported each gender. Round each answer to one decimal place.
2. Do your three answers in #1 add to 100%? If not, why might that be?
3. Determine the number of people in your school or community. If the percentage of people who reported each gender was the same as in #1, how many people of each gender would be in your school or community?
4. Look for recent Statistics Canada census data about gender. Has the percentage of people reporting as non-binary gone up or down since 2021? Why do you think that might be?

Solution to Counting People by Gender

1. The total number of people is $14,842,10 + 15,452,645 + 41,350 = 30,336,135$.

Gender	Number of people	Percentage
men	14,842,140	$\frac{14,842,140}{30,336,135} \times 100\% \approx 48.9\%$
women	15,452,645	$\frac{15,452,645}{30,336,135} \times 100\% \approx 50.9\%$
non-binary people	41,350	$\frac{41,350}{30,336,135} \times 100\% \approx 0.1\%$
Total	**30,336,135**	**99.9%**

2. The calculated percentages add to $48.9 + 50.9 + 0.1 = 99.9\%$, and this is because of round-off error when we calculated the percentages to one decimal place.

3. The number of students at the University of Waterloo in 2021 was 42,000. If the percentage of people of each gender was the same as in #1, the number of people of each gender at the University of Waterloo would be as follows:

Gender	Percentage	Number of people at UW
men	48.9%	$\frac{48.9}{100} \times 42,000 \approx 20,538$
women	50.9%	$\frac{50.9}{100} \times 42,000 \approx 21,378$
non-binary people	0.1%	$\frac{0.1}{100} \times 42,000 \approx 42$

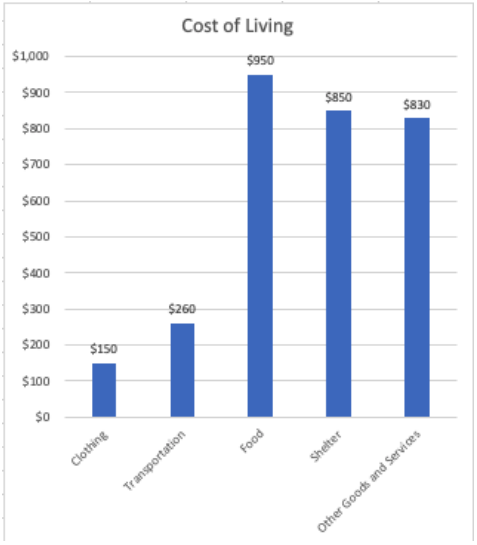
Source: https://uwaterloo.ca/about/sites/default/files/uploads/documents/fp1330_facts_and_figures_2021_lr_final-ua.pdf

We rounded each answer to a whole number.

4. This data is current as of Fall 2023. Once the census has happened a few more times, it will be possible to see how gender distribution changes over time.

Shelter

In order for a country to try to reduce poverty, it is helpful to understand the cost of the basic necessities of life. The bar graph below shows the typical monthly Cost of Living in 2015, according to Statistics Canada.



1. Using the data in the graph, determine the total Cost of Living per month.
2. What percent of the total Cost of Living represented shelter?
3. Do you think the Cost of Living is the same no matter where you live? What places might be more expensive, and why?
4. In Canada, the "Market Basket Measure" keeps track of the typical total Cost of Living in different places, and how it changes over time. Searching online for "Market Basket Measure", can you find more recent costs (for one year) in your region?
   How have they changed since 2015?

Source: https://www.canada.ca/en/employment-social-development/programs/poverty-reduction/reports/strategy.html

Source: https://commons.wikimedia.org/wiki/File:Iqaluit_housing_-d.jpg

Solution to Shelter

1. The total Cost of Living (that is, the cost of these necessities) is $\$150 + \$260 + \$950 + \$850 + \$830 = \$3,040$.

2. Shelter represents $\dfrac{\$850}{\$3,040} \times 100\% \approx 30\%$ of the total Cost of Living.

3. Housing is often more expensive in big cities like Toronto and Vancouver. Heating costs will be higher in the coldest parts of the country. Food may be more expensive in remote places that require more shipping.

4. Waterloo has a population over 500,000. According to Statistics Canada, for Ontario cities of this size, the 2022 Market Basket Measure lists the Cost of Living as $51,648 per year, that is

$$\frac{\$51,648}{12} \approx \$4,289 \text{ per month.}$$

That is significantly higher than in 2015.

As of 2024, inflation has been driving up the cost of living at a much higher rate than it grew in previous decades. Mathematics is used to help work towards providing affordable shelter despite inflation.

Ontario's Pick-2 Lottery

Purchasing lottery tickets is a popular way to gamble. According to the Ontario Lottery and Gaming Corporation, in 2021, 54% of adult Ontarians purchased at least one ticket during the year. The University of Waterloo has a Gambling Research Lab that investigates the complicated relationships between human psychology and the mathematics of gambling.

Source: https://www.olg.ca/en/lottery/play-pick-2/about.html

Do you think it is profitable to play Pick-2, on average? Let's test your guess.

1. In Pick-2, a player chooses a first digit and a second digit for their ticket, with each digit from 0 to 9 inclusive. How many different choices of two-digit numbers are there for a Pick-2 ticket?

2. When a prize draw is performed, a first digit and a second digit are drawn at random, each from 0 to 9 inclusive. There are two ways to win a prize.

Prize Category	Prize
Both digits correct	$99
Only the first digit correct	$2

   (a) What is the probability of getting both digits correct?

   (b) What is the probability of getting only the first digit correct?

3. Suppose that 20,000 tickets are sold, and each possible two-digit number is chosen by the same number of people. For each Prize Category, fill in the number of tickets we would expect to win. Then fill in the total prize money that will be paid out to all of the winners. What is the total of all Prize Money Paid Out?

Prize Category	# Tickets expected to win	Total Prize Money Paid Out
Both digits correct		
Only the first digit correct		
Total		

4. Suppose that the 20,000 tickets were sold for $2 each.

   (a) What is the total revenue from ticket sales?

   (b) What percent of the revenue gets paid out as Prize Money?

5. Given your answers above, does the lottery corporation make profit on the sales of Pick-2 tickets? What does this tell us about whether it is profitable for a person to play Pick-2?

Solution to Lottery Tickets and Gambling

1. Since there are 10 choices for the first digit and 10 choices for the second digit, the number of choices is $10 \times 10 = 100$.

2. (a) The probability of getting the first digit correct is $\frac{1}{10}$ and the probability of getting the second digit correct is also $\frac{1}{10}$. Thus the probability of getting both digits correct is $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$.

   (b) As we saw, the probability of getting the first digit correct is $\frac{1}{10}$. The probability of getting the second digit incorrect is $1 - \frac{1}{10} = \frac{9}{10}$. Thus the probability of getting only the first digit correct (that is, the first digit correct and the second digit incorrect) is $\frac{1}{10} \times \frac{9}{10} = \frac{9}{100}$.

3. Since each possible two-digit number is chosen by the same number of people, we expect $\frac{1}{100}$ of the tickets to have both digits correct, and $\frac{9}{100}$ of the tickets to have only the 1st digit correct.

Prize Category	**# Tickets expected to win**	**Total Prize Money Paid Out**
Both digits correct	$\frac{1}{100} \times 20,000 = 200$	$200 \times \$99 = \$19,800$
Only the first digit correct	$\frac{9}{100} \times 20,000 = 1,800$	$1,800 \times \$2 = \$3,600$
Total	$200 + 1,800 = 2,000$	$\$19,800 + \$3,600 = \$23,400$

4. (a) The total revenue is $20,000 \times \$2 = \$40,000$.

   (b) The percentage of revenue that gets paid out as Prize Money is $\dfrac{\$23,400}{\$40,000} \times 100\% = 58.5\%$.

5. Since only a portion (58.5%) of the ticket revenue is paid out as prize money, that means that on average, people who play Pick-2 are losing more money than they win. In general, it is not profitable to play a lottery.

Some interesting questions to consider:

- Is it a reasonable assumption that every possible two-digit number is chosen by the same number of people?
- If not, how could it change what percent of ticket revenue gets paid out in prizes?
- How does a lottery corporation ensure that they will always make a profit?
- Lotteries and other forms of gambling generate lots of profits. Some of that money is donated to charitable causes. People who play the lottery lose money on average, and some people lose a lot of money. What are the advantages and disadvantages?

Cree Word Tree Puzzle

Cree is a group of Indigenous languages spoken by about 100,000 people. Watch a 2-minute YouTube video to hear five words in Cree.

Dr. Edward Doolittle studies word patterns in Cree, and uses mathematics and computer science to create Cree word puzzles.

Source: https://www.giftoflanguageandculture.ca/giftoflanguageandculture-ca-cree-language/ and https://www.fnuniv.ca/academic/faculty/dr-edward-doolittle/

paspinēw
paskinēw
pāskinēw
tāskinēw
tāhkinēw

A **Word Tree** is a list of words where each word differs from the previous word by just one letter.

In the word tree to the left, notice that the Cree letter a is different from the letter ā. Similarly, the Cree letter i differs from ī, o differs from ō and so on.

1. In the word tree to the right, colour each letter that is different as you move down the list.

tāwinam
tawinam
kawinam
pawinam
pīwinam

2. The word tree to the right starts with the word "nakinam" and ends with the word "sikosam". Fill in the word tree with the following five words, colouring each letter that is different as you move down the list.

   sīkonam sakinam sikonam sīkinam sēkonam

nakinam
_____
_____
_____
_____
_____
sikosam

3. The word tree to the right starts with "nikamow", ends with "wākamon" and has six other words in between.

   Create a new, shorter word tree that

   - starts with "nikamow",
   - ends with "wākamon",
   - includes only words from the original word tree,
   - and has only **four** other words in between.

nikamow
nakamow
sakamow
sākamow
wākamow
wāpamow
wāpamon
wākamon

nikamow
_____
_____
_____
_____
wākamon

Solution to Cree Word Tree Puzzle

There are two ways to write Cree words. This question uses **Standard Roman Orthography**, where letters from the Roman alphabet can include accents, such as ā. We also show each word using **Cree Syllabics**, which builds words from its own set of characters.

1.

Word Tree	Meaning	Cree Syllabics
tāwinam	s/he encounters it	ᑖᐃᓇᒪᒼ
tawinam	s/he opens it	ᑕᐃᓇᒪᒼ
kawinam	s/he breaks it	ᑲᐃᓇᒪᒼ
pawinam	s/he shakes it out	ᐸᐃᓇᒪᒼ
pīwinam	s/he scatters it	ᐱᐃᓇᒪᒼ

2.

Word Tree	Meaning	Cree Syllabics
nakinam	s/he stops it	ᓇᑭᓇᒪᒼ
sakinam	s/he holds it tightly	ᓴᑭᓇᒪᒼ
sīkinam	s/he pours it out	ᓰᑭᓇᒪᒼ
sīkonam	s/he empties it	ᓰᑯᓇᒪᒼ
sēkonam	s/he puts it under	ᓵᑯᓇᒪᒼ
sikonam	s/he crushes it	ᓯᑯᓇᒪᒼ
sikosam	s/he chops it small	ᓯᑯᓴᒪᒼ

3.

Word Tree	Meaning	Cree Syllabics
nikamow	s/he sings	ᓂᑲᒪᐤ
nakamow	s/he sings	ᓇᑲᒪᐤ
sakamow	s/he holds on	ᓴᑲᒪᐤ
sākamow	s/he sticks out	ᓵᑲᒪᐤ
wākamow	it runs crooked	ᐋᑲᒪᐤ
wākamon	it has curves	ᐋᑲᒪᐣ

Sources: Dr. E. Doolittle's talk http://www.birs.ca/events/2012/5-day-workshops/12w5076/videos/watch/201211210914-Doolittle.html at 1:10:00, and some translations from the Online Cree Dictionary. Thank you to Dr. E. Doolittle for sharing Cree language word searches and resources, and for providing advice on this topic.

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